10

Exterior powers and Clifford algebras

In this chapter, various algebraic constructions (exterior products and Clifford algebras) are used to embed some geometries related to projective and polar spaces (subspace and spinor geometries) into projective spaces. In the process, we learn more about the geometries themselves.

10.1 Tensor and exterior products

Throughout this chapter, F is a commutative field (except for a brief discussion of why this assumption is necessary).

The *tensor product* $V \otimes W$ of two F-vector spaces V and W is the free-est bilinear product of V and W: that is, if (as customary), we write the product of vectors $\mathbf{v} \in V$ and $\mathbf{w} \in W$ as $\mathbf{v} \otimes \mathbf{w}$, then we have

$$(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}, \qquad (\alpha\mathbf{v}) \otimes \mathbf{w} = \alpha(\mathbf{v} \otimes \mathbf{w}),$$
$$\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2, \qquad \mathbf{v} \otimes (\alpha\mathbf{w}) = \alpha(\mathbf{v} \otimes \mathbf{w}).$$

Formally, we let X be the F-vector space with basis consisting of all the ordered pairs $(\mathbf{v}, \mathbf{w})$ $(\mathbf{v} \in V, \mathbf{w} \in W)$, and Y the subspace spanned by all expressions of the form $(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) - (\mathbf{v}_1, \mathbf{w}) - (\mathbf{v}_2, \mathbf{w})$ and three similar expressions; then $V \otimes W = X/Y$, with $\mathbf{v} \otimes \mathbf{w}$ the image of $(\mathbf{v}, \mathbf{w})$ under the canonical projection. Sometimes, to emphasize the field, we write $V \otimes_F W$.

This construction will only work as intended over a commutative field. For

$$\alpha\beta(\mathbf{v} \otimes \mathbf{w}) = \alpha(\beta\mathbf{v} \otimes \mathbf{w}) = \beta\mathbf{v} \otimes \alpha\mathbf{w} = \beta(\mathbf{v} \otimes \alpha\mathbf{w}) = \beta\alpha(\mathbf{v} \otimes \mathbf{w}),$$

so if $\mathbf{v}\otimes\mathbf{w}\neq 0$ then $\alpha\beta=\beta\alpha$.

There are two representations convenient for calculation. If V has a basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and W a basis $\{\mathbf{w}_1,\ldots,\mathbf{w}_m\}$, then $V\otimes W$ has a basis

$$\{\mathbf{v}_i\otimes\mathbf{w}_j : 1\leq i\leq n, 1\leq j\leq m\}.$$

If V and W are identified with F^n and F^m respectively, then $V\otimes W$ can be identified with the space of $n\times m$ matrices over F, where $\mathbf{v}\otimes\mathbf{w}$ is mapped to the matrix $\mathbf{v}^\top\mathbf{w}$.

In particular, $\mathrm{rk}(V\otimes W)=\mathrm{rk}(V)\cdot\mathrm{rk}(W)$.

Suppose that V and W are F-algebras (that is, have an associative multiplication which is compatible with the vector space structure). Then $V\otimes W$ is an algebra, with the rule

$$(\mathbf{v}_1\otimes\mathbf{w}_1)\cdot(\mathbf{v}_2\otimes\mathbf{w}_2)=(\mathbf{v}_1\cdot\mathbf{v}_2)\otimes(\mathbf{w}_1\cdot\mathbf{w}_2).$$

Of course, we can form the tensor product of a space with itself; and we can form iterated tensor products of more than two spaces. Let $\bigotimes^k V$ denote the k-fold tensor power of V. Now the *tensor algebra* of V is defined to be

$$T(V)=\bigoplus_{k=0}^{\infty}(\bigotimes^k V),$$

with multiplication given by the rule

$$(\mathbf{v}_1\otimes\ldots\otimes\mathbf{v}_n)\cdot(\mathbf{v}_{n+1}\otimes\ldots\otimes\mathbf{v}_{m+n})=\mathbf{v}_1\otimes\ldots\otimes\mathbf{v}_{m+n}$$

on homogeneous elements, and extended linearly. It is the free-est associative algebra generated by V.

The *exterior square* of a vector space V is the free-est bilinear square of V in which the square of any element of V is zero. In other words, it is the quotient of $\bigotimes^2 V$ by the subspace generated by all vectors $\mathbf{v}\otimes\mathbf{v}$ for $\mathbf{v}\in V$. We write it as $\bigwedge^2 V$, or $V\wedge V$, and denote the product of $\mathbf{v}$ and $\mathbf{w}$ by $\mathbf{v}\wedge\mathbf{w}$. Note that $\mathbf{w}\wedge\mathbf{v}=-\mathbf{v}\wedge\mathbf{w}$. If $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is a basis for V, then a basis for $V\wedge V$ consists of all vectors $\mathbf{v}_1\wedge\mathbf{v}_j$, for $1\leq i<j\leq n$; so

$$\mathrm{rk}(V\wedge V)=\binom{n}{2}=\tfrac{1}{2}n(n-1).$$

More generally, we can define the k^{th} *exterior power* $\bigwedge^k V$ as a k-fold multilinear product, in which any product of vectors vanishes if two factors are equal.

Its basis consists of all expressions $\mathbf{v}_{i_1} \wedge \ldots \wedge \mathbf{v}_{i_k}$, with $1 \leq i_1 < \ldots < i_k \leq n$; and its dimension is $\binom{n}{k}$. Note that $\bigwedge^k V = 0$ if $k > n = \mathrm{rk}(V)$.

The *exterior algebra* of V is

$$\bigwedge(V) = \bigoplus_{k=0}^{n} (\bigwedge^k V),$$

with multiplication defined as for the tensor algebra. Its rank is $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

If θ is a linear transformation on V, then θ induces in a natural way linear transformations $\bigotimes^k \theta$ on $\bigotimes^k V$, and $\bigwedge^k \theta$ on $\bigwedge^k V$, for all k. If $\mathrm{rk}(V) = n$, then we have $\mathrm{rk}(\bigwedge^n V) = 1$, and so $\bigwedge^n \theta$ is a scalar. In fact, $\bigwedge^n \theta = \det(\theta)$. (This fact is the basis of an abstract, matrix-free, definition of the determinant.)

Exercises

1. Let F be a skew field, V a right F-vector space, and W a left vector space. Show that it is possible to define $V \otimes_F W$ as an abelian group so that

$$(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}, \quad \mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2$$

and

$$(\mathbf{v}\alpha) \otimes \mathbf{w} = \mathbf{v} \otimes (\alpha \mathbf{w}).$$

2. In the identification of $F^n \otimes F^m$ with the space of $n \times m$ matrices, show that the rank of a matrix is equal to the minimum r for which the corresponding tensor can be expressed in the form $\sum_{i=1}^{r} \mathbf{v}_i \otimes \mathbf{w}_i$. Show that, in such a minimal expression, $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent, as are $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

3. (a) If K is an extension field of F, and n a positive integer, prove that

$$M_n(F) \otimes_F K \cong M_n(K),$$

where $M_n(F)$ is the ring of $n \times n$ matrices over F.

(b) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

4. Define the *symmetric square* S^2V of a vector space V, the free-est bilinear square of V in which $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$. Find a basis for it, and calculate its dimension. More generally, define the k^{th} *symmetric power* S^kV, and calculate its dimension; and define the *symmetric algebra* $S(V)$. If $\dim(V) = n$, show that the symmetric algebra on V is isomorphic to the polynomial ring in n variables over the base field.

5. Prove that, if θ is a linear map on V, where $\mathrm{rk}(V) = n$, then $\bigwedge^n \theta = \det(\theta)$.

10.2 The geometry of exterior powers

Let V be an F-vector space of rank n, and k a positive integer less than n. There are a couple of ways of defining a geometry on the set $\Sigma_k = \Sigma_k(V)$ of subspaces of V of rank k (equivalently, the $(k-1)$-dimensional subspaces of $\mathrm{PG}(\binom{n}{k}-1,F)$, which I now describe.

The first approach produces a point-line geometry. For each pair U_1,U_2 of subspaces of V with $U_1 \subset U_2$, $\mathrm{rk}(U_1) = k-1$, $\mathrm{rk}(U_2) = k+1$, a *line*

$$L(U_1,U_2) = \{W \in \Sigma_k : U_1 \subset W \subset U_2\}.$$

Now two points lie in at most one line. For, if W_1,W_2 are distinct subspaces of rank k and $W_1,W_2 \in L(U_1,U_2)$, then $U_1 \subseteq W_1 \cap W_2$ and $\langle W_1,W_2\rangle \subseteq U_2$; so equality must hold in both places. Note that two subspaces are collinear if and only if their intersection has codimension 1 in each. We call this geometry a *subspace geometry*.

In the case $k = 2$, the points of the subspace geometry are the lines of $\mathrm{PG}(n-1,F)$, and its lines are the plane pencils. In particular, for $k = 2$, $n = 4$, it is the Klein quadric.

The subspace geometry has the following important property:

Proposition 10.1 *If three points are pairwise collinear, then they are contained in a projective plane. In particular, a point not on a line L is collinear with none, one or all points of L.*

Proof Clearly the second assertion follows from the first. In order to prove the first assertion, note that there are two kinds of projective planes in the geometry, consisting of all points W (i.e., subspaces of rank k) satisfying $U_1 \subset W \subset U_2$, where either $\mathrm{rk}(U_1) = k-1$, $\mathrm{rk}(U_2) = k+2$, or $\mathrm{rk}(U_1) = k-2$, $\mathrm{rk}(U_2) = k+1$.

So let W_1,W_2,W_3 be pairwise collinear points. If $\mathrm{rk}(W_1 \cap W_2 \cap W_3) = k-1$, then the three points are contained in a plane of the first type; so suppose not. Then we have $\mathrm{rk}(W_1 \cap W_2 \cap W_3) = k-2$; and, by factoring out this intersection, we may assume that $k = 2$. In the projective space, W_1,W_2,W_3 are now three pairwise intersecting lines, and so are coplanar. Thus $\mathrm{rk}\langle W_1,W_2,W_3\rangle = k+1$, and our three points lie in a plane of the second type. ■

A point-line geometry satisfying the second conclusion of Proposition 10.1 is called a *gamma space*. Gamma spaces are a natural generalisation of polar spaces (in the Buekenhout–Shult sense), and this property has been used in several recent characterisations (some of which are surveyed by Shult [29]).

The subspace geometries have natural embeddings in projective spaces given by exterior powers, generalising the Klein quadric. Let $X = \bigwedge^k V$; we consider the projective space $\mathrm{PG}(N-1,F)$ based on X, where $N = \binom{n}{k}$. This projective space contains some distinguished points, those spanned by the vectors of the form $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k$, for $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$. We call these *pure products*.

Theorem 10.2 *(a)* $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k = 0$ *if and only if* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are linearly dependent.*

(b) The set of points of $\mathrm{PG}(N-1,F)$ *spanned by non-zero pure products, together with the lines meeting this set in more than two points, is isomorphic to the subspace geometry* $\Sigma_k(V)$.

Proof (a) If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, then they form part of a basis, and their product is one of the basis vectors of X, hence non-zero. Conversely, if these vectors are dependent, then one of them can be expressed in terms of the others, and the product is zero (using linearity and the fact that a product with two equal terms is zero).

(b) It follows from our remarks about determinants that, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are replaced by another k-tuple with the same span, then $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k$ is multiplied by a scalar factor, and the point of $\mathrm{PG}(N-1,F)$ it spans is unaltered. If $W_1 \neq W_2$, then we can (as usual in linear algebra) choose a basis for V containing bases for both W_1 and W_2; the corresponding pure products are distinct basis vectors of X, and so span distinct points. The correspondence is one-to-one.

Suppose that W_1 and W_2 are collinear in the subspace geometry. then they have bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{w}_1\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{w}_2\}$. Then the points spanned by the vectors

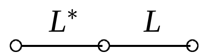
$$\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_{k-1} \wedge (\alpha \mathbf{w}_1 + \beta \mathbf{w}_2)$$

form a line in $\mathrm{PG}(N-1,F)$ and represent all the points of the line in the subspace geometry joining W_1 and W_2.

Conversely, suppose that $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k$ and $\mathbf{w}_1 \wedge \ldots \wedge \mathbf{w}_k$ are two pure products. By factoring out the intersection of the corresponding subspaces, we may assume that $\mathbf{v}_1, \ldots, \mathbf{w}_k$ are linearly independent. If $k > 1$, then no other vector in the span of these two pure products is a pure product. If $k = 1$, then the three points are coplanar. ∎

The other natural geometry on the set $\Sigma_k(V)$ is just the truncation of the projective geometry to ranks $k-1, k$ and $k+1$; in other words, its varieties are the subspaces of V of these three ranks, and incidence is inclusion. This geometry has no immediate connection with exterior algebra; but it (or the more general form based on any generalised projective geometry) has a beautiful characterisation due to Sprague (1981).

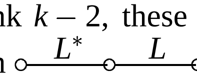
Theorem 10.3 *(a) The geometry just described has diagram*



where L^* denotes the class of dual linear spaces.

(b) Conversely, any geometry with this diagram, in which chains of subspaces are finite, consists of the varieties of ranks $k-1, k$ and $k+1$ of a generalised projective space of finite dimension, two varieties incident if one contains the other.

Proof The residue of a variety of rank $k-1$ is the quotient projective space; and the residue of a variety of rank $k+1$ is the dual of $\mathrm{PG}(k,F)$. This establishes the diagram.

I will not give the proof of Sprague's theorem. the proof is by induction (hence the need to assume finite rank). Sprague shows that it is possible to recognise in the geometry objects corresponding to varieties of rank $k-2$, these objects together with the left and centre nodes forming the diagram  again, but with the dimension of the residue of a variety belonging to the rightmost node reduced by 1. After finitely many steps, we reach the points, lines and planes of the projective space, which is recognised by the Veblen–Young axioms. ■

Exercise

1. Show that the dual of the generalised hexagon $G_2(F)$ constructed in Section 8.8 is embedded in the subspace geometry of lines of $\mathrm{PG}(6,F)$. [Hint: the lines of the hexagon through a point x are all those containing x in a plane $W(x)$.]

10.3 Near polygons

In this section we consider certain special point-line geometries. These geometries will always be connected, and the *distance* between two points is the smallest number of lines in a path joining them. A *near polygon* is a geometry with the following property:

(NP) Given any point p and line L, there is a unique point of L nearest to p.

If a near polygon has diameter n, it is called a *near* $2n$*-gon*.

We begin with some elementary properties of near polygons.

Proposition 10.4 *In a near polygon,*

(a) two points lie on at most one line;

(b) the shortest circuit has even length.

Proof (a) Suppose that lines L_1, L_2 contain points p_1, p_2. Let $q \in L_1$. Then q is at distance 1 from the two points p_1, p_2 of L_2, and so is at distance 0 from a unique point of L_2; that is, $q \in L_2$. So $L_1 \subseteq L_2$; and, interchanging these two lines, we find that $L_1 = L_2$.

If a circuit has odd length $2m+1$, then a point lies at distance m from two points of the opposite line; so it lies at distance $m-1$ from some point of this line, and a circuit of length $2m$ is formed. ■

Any generalised polygon is a near polygon; and any "non-degenerate" near 4-gon is a generalised quadrangle (see Exercise 1).

Some deeper structural properties are given in the next two theorems, which were found by Shult and Yanushka [30].

Theorem 10.5 *Suppose that* $x_1x_2x_3x_4$ *is a circuit of length 4 in a near polygon, at least one of whose sides contains more than two points. Then there is a unique subspace containing these four points which is a generalised quadrangle.* ■

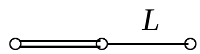
A subspace of the type given by this theorem is called a *quad*.

Corollary 10.6 *Suppose that a near polygon has the properties*

(a) any line contains more than two points;

(b) any two points at distance 2 are contained in a circuit of length 4.

Then the points, lines and quads form a geometry belonging to the diagram



.

We now assume that the hypotheses of this Corollary apply. Let p be a point and Q a quad. We say that the pair (p,Q) is *classical* if

(a) there is a unique point x of Q nearest p;

(b) for $y \in Q$, $d(y,p) = d(x,p)+1$ if and only if y is collinear with x.

(The point x is the "gateway" to Q from p.) An *ovoid* in a generalised quadrangle is a set O of (pairwise non-collinear) points with the property that any further point of the quadrangle is collinear with a unique point of O. The point-quad pair (p,Q) is *ovoidal* if the set of points of Q nearest to p is an ovoid of Q.

Theorem 10.7 *In a near polygon with at least three points on a line, any point-quad pair is either classical or ovoidal.* ■

A proof in the finite case is outlined in Exercise 2.

We now give an example, the sextet geometry of Section 9.3 (which, as we already know, has the correct diagram). Recall that the POINTs, LINEs, and "QUADs" (as we will now re-name them) of the geometry are the octads, trios and sextets of the Witt system. We check that this is a near polygon, and examine the point-quad pairs.

Two octads intersect in 0, 2 or 4 points. If they are disjoint, they are contained in a trio (i.e., collinear). If they intersect in four points, they define a sextet, and so some octad is disjoint from both; so their distance is 2. If they intersect in two points, their distance is 3. Suppose that $\{B_1,B_2,B_3\}$ is a trio and B an octad not in this trio. Either B is disjoint from (i.e., collinear with) a unique octad in the trio, or its intersections with them have cardinalities 4, 2, 2. In the latter case, it lies at distance 2 from one POINT of the LINE, and distance 3 from the other two.

Now let B be a POINT (an octad), and S a QUAD (a sextet). The intersections of B with the tetrads of S have the property that any two of them sum to 0, 2, 4 or 8; so they are all congruent mod 2. If the intersections have even parity, they are $4,4,0,0,0,0$ (the POINT lies in the QUAD) or $2,2,2,2,0,0$ (B is disjoint from a unique octad incident with S, and the pair is classical). If they have odd parity, they are $3,1,1,1,1,1$; then B has distance 2 from the five octads containing the first tetrad, and distance 3 from the others. Note that in the GQ of order $(2,2)$, represented as the pairs from a 6-set, the five pairs containing an element of the 6-set form an ovoid. So (B,S) is ovoidal in this case.

10.3.1 Exercises

1. (a) A near polygon with lines of size 2 is a bipartite graph.

(b) A near 4-gon, in which no point is joined to all others, is a generalised quadrangle.

2. Let Q be a finite GQ with order s,t, where $s > 1$.

(a) Suppose that the point set of Q is partitioned into three subsets A,B,C such that for any line L, the values of $|L \cap A|$, $|L \cap B|$ and $|L \cap C|$ are either $1,s,0$, or $0,1,s$. Prove that A is a singleton, and B the set of points collinear with A.

(b) Suppose that the point set of Q is partitioned into two subsets A and B such that any line contains a unique point of A. Prove that A is an ovoid.

(c) Hence prove (10.3.4) in the finite case.

10.4 Dual polar spaces

We now look at polar spaces "the other way up". That is, given an abstract polar space of polar rank n, we consider the geometry whose POINTs and LINEs are the subspaces of dimension $n-1$ and $n-2$ respectively, incidence being reversed inclusion. (This geometry was introduced in Section 7.4.)

Proposition 10.8 *A dual polar space of rank n is a near* $2n$*-gon.*

Proof This is implicit in what we proved in Proposition 7.9. ■

Any dual polar space has girth 4, and any circuit of length 4 is contained in a unique quad. Moreover, the point-quad pairs are all classical. Both these assertions are easily checked in the polar space by factoring out the intersection of the subspaces in question.

The converse of this result was proved by Cameron [9]. It is stated here using the notation and ideas (and simplifications) of Shult and Yanushka described in the last section.

Theorem 10.9 *Let* $\mathcal{G}$ *be a near* $2n$*-gon. Suppose that* *(a) any 4-circuit is contained in a quad;*

(b) any point-quad pair is classical;

(c) chains of subspaces are finite.

Then $\mathcal{G}$ is a dual polar space of rank n.

Proof The ideas behind the proof will be sketched.

Given a point p, the residue of p (that is, the geometry of lines and quads containing p) is a linear space, by hypothesis (a). Using (b), it is possible to show that this linear space satisfies the Veblen–Young axioms, and so is a projective space $\mathcal{P}(p)$ (possibly infinite-dimensional). We may assume that this geometry has dimension greater than 2 (otherwise the next few steps are vacuous).

Now, given points p and q, let $\mathcal{X}(p,q)$ be the set of lines through p (i.e., points of $\mathcal{P}(p)$) which belong to geodesics from p to q (that is, which contain points r with $d(q,r) = d(p,q) - 1$). This set is a subspace of $\mathcal{P}(p)$. Let X be any subspace of $\mathcal{P}(p)$, and let

$$\mathcal{Y}(p,X) = \{q : \mathcal{X}(p,q) \subseteq X\}.$$

It can be shown that $\mathcal{Y}(p,X)$ is a subspace of the geometry, containing all geodesics between any two of its points, and that, if p' is any point of $\mathcal{Y}(p,X)$, then there is a subspace X' of $\mathcal{P}(p')$ such that $\mathcal{Y}(p',X') = \mathcal{Y}(p,X)$.

For the final step, it is shown that the subspaces $\mathcal{Y}(p,X)$, ordered by reverse inclusion, satisfy the axioms (P1)–(P4) of Tits. ■

Remark In the case when any line has more than two points, condition (a) is a consequence of (10.3.2), and (10.3.4) shows that (b) is equivalent to the assertion that no point-quad pairs are ovoidal.

10.5 Clifford algebras and spinors

Spinors provide projective embeddings of some geometries related to dual polar spaces, much as exterior powers do for subspace geometries. But they are somewhat elusive, and we have to construct them via Clifford algebras.

Let V be a vector space over a commutative field F, and f a quadratic form on V; let b be the bilinear form obtained by polarising f. The *Clifford algebra* $C(f)$ of f (or of the pair (V, f)) is the free-est algebra generated by V subject to the condition that $\mathbf{v}^2 = f(\mathbf{v})\cdot 1$ for all $\mathbf{v}\in V$. In other words, it is the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements $\mathbf{v}^2 - f(\mathbf{v})\cdot 1$ for $\mathbf{v}\in V$.

Note that $\mathbf{vw}+\mathbf{wv} = b(\mathbf{v},\mathbf{w})\cdot 1$ for $\mathbf{v},\mathbf{w}\in V$.

The Clifford algebra is a generalisation of the exterior algebra, to which it reduces if f is identically zero. And it has the same dimension:

Proposition 10.10 *Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a basis for V. Then $C(f)$ has a basis consisting of all vectors $\mathbf{v}_{i_1}\cdots\mathbf{v}_{i_k}$, for $0\le i_1<\ldots<i_k\le n$; and so $\mathrm{rk}(C(f)) = 2^n$.*

Proof Any product of basis vectors can be rearranged into non-decreasing order, modulo products of smaller numbers of basis vectors, using

$$\mathbf{wv} = \mathbf{vw} - b(\mathbf{v},\mathbf{w})\cdot 1.$$

A product with two terms equal can have its length reduced. Now the result follows by multilinearity. ■

In an important special case, we can describe the structure of $C(f)$.

Theorem 10.11 *Let f be a split quadratic form of rank n over F (equivalent to*

$$x_1x_2+x_3x_4+\ldots+x_{2n-1}x_{2n}.$$

Then $C(f)\cong M_{2^n}(F)$, the algebra of $2^n\times 2^n$ matrices over F.

Proof It suffices to find a linear map $\theta: V\to M_{2^n}(F)$ satisfying

(a) $\theta(V)$ generates $M_{2^n}(F)$ (as algebra with 1);

(b) $\theta(\mathbf{v})^2 = f(\mathbf{v})I$ for all $\mathbf{v}\in V$.

For if so, then $M_{2^n}(F)$ is a homomorphic image of $C(f)$; comparing dimensions, they are equal.

We use induction on n. For $n=0$, the result is trivial. Suppose that it is true for n, with a map θ. Let $\tilde{V} = V\perp\langle\mathbf{x},\mathbf{y}\rangle$, where $f(\lambda\mathbf{x}+\mu\mathbf{y}) = \lambda\mu$. Define $\tilde{\theta}:\tilde{V}\to M_{2^{n+1}}(F)$ by

$$\tilde{\theta}(\mathbf{v}) = \begin{pmatrix}\theta(\mathbf{v}) & O\\ O & -\theta(\mathbf{v})\end{pmatrix},\quad \mathbf{v}\in V,$$

$$\tilde{\theta}(\mathbf{x}) = \begin{pmatrix} O & I \\ O & O \end{pmatrix}, \qquad \tilde{\theta}(\mathbf{y}) = \begin{pmatrix} O & O \\ I & O \end{pmatrix},$$

extended linearly.

To show generation, let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2^{n+1}}(F)$ be given. We may assume inductively that A, B, C, D are linear combinations of products of $\theta(\mathbf{v})$, with $\mathbf{v} \in V$. The same combinations of products of $\tilde{\theta}(\mathbf{v})$ have the forms $\tilde{A} = \left(\begin{smallmatrix} A & O \\ O & A^* \end{smallmatrix}\right)$, etc. Now

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \tilde{A}\tilde{\theta}(\mathbf{x})\tilde{\theta}(\mathbf{y}) + \tilde{B}\tilde{\theta}(\mathbf{x}) + \tilde{\theta}(\mathbf{y})\tilde{C} + \tilde{\theta}(\mathbf{y})\tilde{D}\tilde{\theta}(\mathbf{x}).$$

To establish the relations, we note that

$$\tilde{\theta}(\mathbf{v} + \lambda\mathbf{x} + \mu\mathbf{y}) = \begin{pmatrix} \theta(\mathbf{v}) & \lambda I \\ \mu I & -\theta(\mathbf{v}) \end{pmatrix},$$

and the square of the right-hand side is $(f(\mathbf{v}) + \lambda\mu)\left(\begin{smallmatrix} I & O \\ O & I \end{smallmatrix}\right)$, as required. ∎

More generally, the argument shows the following.

Theorem 10.12 *If the quadratic form f has rank n and germ f_0, then*

$$C(f) \cong C(f_0) \otimes_F M_{2^n}(F).$$

∎

In particular, $C(x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n})$ is the direct sum of two copies of $M_{2^n}(F)$; and, if α is a non- square in F, then

$$C(\alpha x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}) \cong M_{2^n}(K),$$

where $K = F(\sqrt{\alpha})$.

Looked at more abstractly, Theorem 10.12 says that the Clifford algebra of the split form of rank n is isomorphic to the algebra of endomorphisms of a vector space S of rank 2^n. This space is the *spinor space*, and its elements are called *spinors*. Note that the connection between the spinor space and the original vector space is somewhat abstract and tenuous! It is the spinor space which carries the geometrical structures we now investigate.

Exercise

1. Prove that the Clifford algebras of the real quadratic forms $-x^2$ and $-x^2 - y^2$ respectively are isomorphic to the complex numbers and the quaternions. What is the Clifford algebra of $-x^2 - y^2 - z^2$?

10.6 The geometry of spinors

In order to connect spinors to the geometry of the quadratic form, we first need to recognise the points of a vector space within its algebra of endomorphisms.

let V be a vector space, A the algebra of linear transformations of V. Then A is a simple algebra. If U is any subspace of V, then

$$I(U) = \{a \in A : \mathbf{v}a \in U \text{ for all } \mathbf{v} \in V\}$$

is a left ideal in A. Every left ideal is of this form (see Exercise 1). So the projective space based on V is isomorphic to the lattice of left ideals of A. In particular, the minimal left ideals correspond to the points of the projective space. Moreover, if U has rank 1, then $I(U)$ has rank n, and A (acting by left multiplication) induces the algebra of linear transformations of U. In this way, the vector space is "internalised" in the algebra.

Now let V carry a split quadratic form of rank n. If U is a totally singular subspace of rank n, then the elements of U generate a subalgebra isomorphic to the exterior algebra of U. Let $\hat{U}$ denote the product of the vectors in a basis of U. Note that $\hat{U}$ is unchanged, apart from a scalar factor, if a different basis is used. Then $\mathbf{vu}\hat{U} = 0$ whenever $\mathbf{v} \in V$, $\mathbf{u} \in U$, and $\mathbf{u} \neq 0$; so the left ideal generated by $\hat{U}$ has dimension 2^n (with a basis of the form $\{\mathbf{v}_{i_1} \dots \mathbf{v}_{i_k} \hat{U}\}$, where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of a complement for U, and $1 \leq i_1 < \dots < i_k \leq n$. Thus, $\hat{U}$ generates a minimal left ideal of $C(f)$. By the preceding paragraph, this ideal corresponds to a point of the projective space $\mathrm{PG}(2^n - 1, F)$ based on the spinor space S.

Summarising, we have a map from the maximal totally singular subspaces of the hyperbolic quadric to a subset of the points of projective spinor space. The elements in the image of this map are called *pure spinors*.

We now state some properties of pure spinors without proof.

Proposition 10.13 *(a) There is a decomposition of the spinor space S into two subspaces S^+, S^-, each of rank 2^{n-1}. Any pure spinor is contained in one of these subspaces.*

(b) Any line of spinor space which contains more than two pure spinors has the form

$$\{\langle \hat{U} \rangle : U \text{ is t.s. with rank } n,\ U \text{ has type } \varepsilon,\ U \supset W\},$$

where W is a t.s. subspace of rank $n - 2$, and $\varepsilon = \pm 1$. ■

In (a), the subspaces S^+ and S^- are called *half-spinor spaces*.

In (b), the type of a maximal t.s. subspace is that described in Section 7.4. The maximal t.s. subspaces containing W form a dual polar space of rank 2, which in this case is simply a complete bipartite graph, the parts of the bipartition being the two types of maximal subspace. Any two subspaces of the same type have intersection with even codimension at most 2, and hence intersect precisely in W.

The dual polar space associated with the split quadratic form has two points per line, and so in general is a bipartite graph. The two parts of the bipartition can be identified with the pure spinors in the two half-spinor spaces. The lines described in (b) within each half-spinor space form a geometry, a so-called *half-spinor geometry*: two pure spinors are collinear in this geometry if and only if they lie at distance 2 in the dual polar space. In general, distances in the half-spinor geometry are those in the dual polar space, halved!

Proposition 10.14 *If p is a point and L a line in a half-spinor geometry, then either there is a unique point of L nearest p, or all points of L are equidistant from p.*

Proof Recall that the line L of the half-spinor geometry is "half" of a complete bipartite graph Q, which is a quad in the dual polar space. If the gateway to Q is on L, it is the point of L nearest to p; if it is on the other side, then all points of L are equidistant from p. ■

The cases $n = 3,4$ give us yet another way of looking at the Klein quadric and triality.

Example $n = 3$. The half-spinor space has rank 4. The diameter of the half-spinor geometry is 1, and so it is a linear space; necessarily $\mathrm{PG}(3,F)$: that is, every spinor in the half-spinor space is pure. Points of this space correspond to one family of maximal subspaces on the Klein quadric.

Example $n = 4$. Now the half-spinor spaces have rank 8, the same as V. The half-spinor space has diameter 2, and (by Proposition 10.14) satisfies the Buekenhout–Shult axiom. But we do not need to use the full classification of polar spaces here, since the geometry is already embedded in $\mathrm{PG}(7,F)$! We conclude that each half-spinor space is isomorphic to the original hyperbolic quadric.

We conclude by embedding a couple more dual polar spaces in projective spaces.

Proposition 10.15 *Let f be a quadratic form of rank $n-1$ on a vector space of rank $2n-1$. Then the dual polar space of F is embedded as all the points and some of the lines of the half-spinor space associated with a split quadratic form of rank n.*

Proof We can regard the given space as of the form $\mathbf{v}^\perp$, where $\mathbf{v}$ is a non-singular vector in a space carrying a split quadratic form of rank n. Now each t.s. subspace of rank $n-1$ for the given form is contained in a unique t.s. space of rank n of each type for the split form; so we have an injection from the given dual polar space to a half-spinor space. The map is onto: for if U is t.s. of rank n, then $U \cap \mathbf{c}^\perp$ has rank $n-1$. A line of the dual polar space consists of all the subspaces containing a fixed t.s. subspace of rank $n-2$, and so translates into a line of the half-spinor space, as required. ■

Proposition 10.16 *Let K be a quadratic extension of F, with Galois automorphism σ. Let V be a vector space of rank $2n$ over K, carrying a non-degenerate σ-Hermitian form b of rank n. Then the dual polar space associated with b is embeddable in a half-spinor geometry over F.*

Proof Let $H(\mathbf{v}) = b(\mathbf{v},\mathbf{v})$. Then $H(v) \in F$ for all $\mathbf{v} \in V$; and H is a quadratic form on the space V_F obtained by restricting scalars to F. (Note that V_F has rank $4n$ over F.) Now any maximal t.i. subspace for b is a maximal t.s. subspace for H of rank $2n$; so H is a split form, and we have an injection from points of the dual unitary space to pure spinors. Moreover, the intersection of any two of these maximal t.s. subspaces has even F-codimension in each; so they all have the same type, and our map goes to points of a half-spinor geometry.

A line of the dual polar space is defined by a t.i. subspace of rank $n-1$ (over K), which is t.s. of rank $2n-2$ over F; so it maps to a line of the half-spinor geometry, as required. ■

In the case $n = 3$, we have the duality between the unitary and non-split orthogonal spaces discussed in Section 8.3.

Exercise

1. (a) Prove that the set of endomorphisms of V with range contained in a subspace U is a left ideal.

(b) Prove that, if T has range U, then any endomorphism whose range is contained in U is a left multiple of T.

(c) Deduce that every left ideal of the endomorphism ring of V is of the form described in (a).