Antiflag-transitive collineation groups revisited

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1 Introduction and background

An *antiflag* in a projective space is a non-incident point-hyperplane pair. A subgroup G of $\Gamma\mathrm{L}(n,q)$ is *antiflag-transitive* if it acts transitively on the set of antiflag of $\mathrm{PG}(n-1,q)$.

In 1979, Cameron and Kantor [2] published a paper determining all antiflag-transitive subgroups of $\Gamma\mathrm{L}(n,q)$. A large part of the motivation was the fact that a group which acts 2-transitively on points is necessarily antiflag-transitive, so that the result included the solution to Wagner's problem of determining the 2-transitive collineation groups. At about the same time, this problem was also settled by Orchel in his London Ph.D. thesis [7]; his approach was also based on antiflag-transitivity, though he did not determine all the antiflag-transitive groups.

The techniques used by Cameron and Kantor were largely geometric (involving translation planes and generalized polygons, among other things). Essentially no group theory beyond Sylow's Theorem and its consequences was used.

Shortly afterwards, the classification of finite simple groups was announced (though there were some problems with the proof, which may now have been resolved). Using this classification, Hering [4] showed that it is possible to determine all subgroups of $\Gamma\mathrm{L}(n,q)$ which act transitively on the points of projective space, a much stronger result.

However, the Cameron–Kantor theorem has been used in various places: by Kantor [6] in determining collineation groups containing a Singer cycle, and by Abhyankar (see [1] and subsequent papers) in showing that various groups are Galois groups in non-zero characteristic, as well as others [3, 9, 10]. So it perhaps retains its interest. One purpose of this note is to give a self-contained account.

Recently, it has been observed that there are two problems with the proof. First, Inglis and Saxl pointed out to us that some groups are missing from the conclusion; and second, a result of Perin [8] is used under weaker assumptions than those used in its proof. So another purpose is to fill these gaps.

The paper of Cameron and Kantor goes on to determine the subgroups of classical groups which act transitively on antiflags of the associated polar space (that is, non-collinear pairs of points). We hope to give a similar treatment of this theorem in a sequel.

2 Statement of the theorems

Theorems I, II and III of [2], corrected by the addition of the imprimitive groups, read as follows. Our notation for the groups $\mathrm{SL}(n,q)$, $\Gamma\mathrm{L}(n,q)$, $\mathrm{Sp}(n,q)$, $G_2(q)$ is standard, and $\Gamma\mathrm{Sp}(n,q)$ denotes the normaliser of $\mathrm{Sp}(n,q)$ in $\Gamma\mathrm{L}(n,q)$.

Theorem 2.1 *Let G be a subgroup of $\Gamma\mathrm{L}(n,q)$ which is 2-transitive on points. Then one of the following holds:*

(a) G contains $\mathrm{SL}(n,q)$;

(b) $n=2$, $q=4$, and G is a group of order 20 or 60 inducing the Frobenius group of order 20 on points;

(c) $n=4$, $q=2$, and G is isomorphic to the alternating group A_7.

Theorem 2.2 *Let G be a subgroup of $\Gamma\mathrm{L}(n,q)$ which is antiflag-transitive and primitive but not 2-transitive on points. Then n is even, and one of the following holds:*

(a) $\mathrm{Sp}(n,q) \leq G \leq \Gamma\mathrm{Sp}(n,q)$;

(b) $n=4$, $q=2$, and G is isomorphic to the alternating group A_6;

(c) $n=6$, q is even, and $G_2(q) \leq G \leq \mathrm{Aut}(G_2(q))$;

(d) $n=6$, $q=2$, and G is isomorphic to $\mathrm{PSU}(3,3^2)$.

Theorem 2.3 *Let G be a subgroup of $\Gamma\mathrm{L}(n,q)$ which is antiflag-transitive and imprimitive on points. Then $q=2$ or $q=4$, and G has a normal subgroup $\mathrm{SL}(n/2,q^2)$, $\mathrm{Sp}(n/2,q^2)$ or $G_2(q^2)$ (with $n=12$ in the last case).*

3 Outline of the proof

The proof of the theorems is by induction. However, the induction is somewhat subtle and it does not seem possible to prove any one of the parts of the theorem separately. We state here the three points at which induction comes into the proof.

Theorem 3.1 *Let $G \leq \Gamma\mathrm{L}(n,q)$ with $n \geq 3$, and suppose that G is 2-transitive on points of $\mathrm{PG}(n-1,q)$. Then, for any hyperplane H, the stabiliser G_H of H is antiflag-transitive on H.*

Theorem 3.2 *Let $G \leq \Gamma\mathrm{Sp}(2m,q)$ with $m \geq 2$, and suppose that G has rank 3 on the points of $\mathrm{PG}(2m-1,q)$. Then, for any point x, the stabiliser G_x is antiflag-transitive on $x^\perp/x$, and is not 2-transitive there if $m \geq 3$.*

Theorem 3.3 *Let $G \leq \Gamma\mathrm{L}(n,q)$ with $n \geq 2$, and suppose that G is imprimitive on the points of $\mathrm{PG}(n-1,q)$. Then $q = 2$ or $q = 4$, and G is an antiflag-transitive subgroup of $\Gamma\mathrm{L}(n/2,q^2)$, acting on $\mathrm{GF}(q)^n$ by restriction of scalars.*

Now we can outline the proof of the main theorems. Let G be an antiflag-transitive group, and suppose that all antiflag-transitive groups of smaller degree are known.

1. If G is 2-transitive then, by Theorem 3.1, G_H acts on H as a known group. So G_H is an extension of a group of collineations fixing a hyperplane pointwise by a known group, and the possibilities for G can be determined.

2. If G is primitive but not 2-transitive, then a purely geometric result (Theorem 6.1) shows that either G is a rank 3 subgroup of $\Gamma\mathrm{Sp}(n,q)$, or G acts distance-transitively on a generalized polygon. In the former case, Theorem 3.2 shows that G_x is an extension of a group of symplectic transvections by a known group. In the latter case, the Feit–Higman theorem shows that the geometry is a generalized hexagon, and G turns out to be a subgroup of $\mathrm{Aut}(G_2(q))$ containing $G_2(q)$.

3. If G is imprimitive, Theorem 3.3 shows that it is isomorphic to a known (primitive) group of smaller degree over a larger field.

4 Orbit theorems

Any subgroup of $\Gamma\mathrm{L}(n,q)$ has equally many orbits on points and hyperplanes of $\mathrm{PG}(n-1,q)$. This fact is crucial to the proof of the inductive steps, so we have given a self-contained proof of this (and more) in an Appendix. To illustrate the application of the technique, we begin with an alternative formulation of antiflag-transitivity.

Theorem 4.1 *The following conditions on a subgroup G of $\Gamma\mathrm{L}(n,q)$ are equivalent:*

(a) G is antiflag-transitive;

(b) G_L is 2-transitive on the points of L for all lines L.

Proof Both statements imply that G is transitive on points; so we assume that this is the case. Now condition (b) is equivalent to the condition that the number of orbits of G_x on lines incident with x is equal to the number of orbits on points different from x. For each such line-orbit splits into a number of point-orbits, and the condition that the orbit of L yields a single point-orbit holds if and only if any two points of L lie in the same orbit of G_x; this is clearly the same as (b).

Now let G_x have $n+1$ point-orbits (including $\{x\}$). Then it also has $n+1$ hyperplane-orbits. If G is antiflag-transitive, then just one of these orbits consists of hyperplanes not incident with x; so G has n orbits on hyperplanes containing x. Hence, considering the projective space based on the quotient V/x (whose points and hyperplanes are the lines and hyperplanes respectively containing x), G_x has n orbits on the lines containing x, whence (b) holds. The converse implication is proved by reversing the argument.

Now we prove the first two inductive theorems. The third, for the imprimitive case, needs different techniques.

Proof of Theorem 3.1 Suppose that G is 2-transitive. Let x be any point. Then G_x has two orbits on points, and hence also two orbits on hyperplanes, which must be the set A of hyperplanes containing x and the set B of hyperplanes not containing x. The fact that B is an orbit shows that G is antiflag-transitive. Also, transitivity on A implies that the stabiliser of a hyperplane H is transitive on the points of H.

But more is true. We have $|A| = (q^{n-1}-1)/(q-1)$ and $|B| = q^{n-1}$. These two numbers are coprime, and so G_x is transitive on $A \times B$. Hence G acts transitively on triples (x, H, H'), where x is a point, H a hyperplane containing x, and H' a hyperplane not containing x. So G_H is antiflag-transitive on H. ■

Proof of Theorem 3.2 Suppose that G is a rank 3 subgroup of $\Gamma\mathrm{Sp}(2m,q)$. Then G preserves a polarity π (an incidence-preserving correspondence between points and hyperplanes), so that $x\pi = x^{\perp}$. Now the other two point-orbits A nd B of G_x consist of the points of $x\pi \setminus \{x\}$ and the points outside $x\pi$, respectively.

If $y \in A$, then $y\pi$ is a hyperplane containing x other than $x\pi$; and if $z \in B$, then $z\pi$ is a hyperplane not containing x. Thus the transitivity of G_x on $B\pi$ shows that G is antiflag-transitive.

Transitivity of G_x on A shows that it is point-transitive on $x^{\perp}/x$. Now $|A| = q(q^{2m-2}-1)/(q-1)$, and $|B| = q^{2m-1}$; so, for $z \in B$, all orbits of G_{xz} on A have size divisible by $(q^{2m-2}-1)/(q-1)$. But this is the cardinality of $x^{\perp} \cap z^{\perp}$, so this set is an orbit.Thus, G is transitive on the set of triples (x,y,z) with $z \notin x^{\perp}$ and $y \in x^{\perp} \cap z^{\perp}$. Looked at another way, G_y is transitive on the set of non-perpendicular pairs of points in $y^{\perp} \setminus \{y\}$. Hence G_y is antiflag-transitive on $y^{\perp}/y$, as required. Clearly it is not 2-transitive, since it preserves the induced symplectic structure. ■

5 The base of the induction

The subgroups of $\Gamma\mathrm{L}(2,q)$ and $\Gamma\mathrm{L}(3,q)$ are explicitly known, so it is just a matter of checking the lists to see which ones are antiflag-transitive. (Note that a subgroup of $\Gamma\mathrm{L}(2,q)$ is antiflag-transitive if and only if it is 2-transitive, since a hyperplane of the projective line is a point.)

Since the lists of subgroups are a bit inaccessible, here is an argument for the case $n = 2$ which minimises the dependence on long lists of groups. We use a result of Dickson:

Theorem 5.1 *Let $G \le \mathrm{GL}(2,p^e)$ be generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Then one of the following holds:*

(a) $G = \mathrm{SL}(2,p^f)$, where $\mathrm{GF}(p)(a) = \mathrm{GF}(p^f)$;

(b) $G = A_5$, where $p = 3$ and e is even;

(c) G is dihedral, with $p = 2$*.*

Now suppose that $G \leq \Gamma\mathrm{L}(2, p^f)$ is 2-transitive on points of the projective line. For any point x, a Sylow p-subgroup of G_x is transitive on the remaining p^f points, and a Sylow p-subgroup Q of $G \cap \mathrm{GL}(2, p^f)$ has index at most f in it. So $Q \neq 1$. Choose a basis so that some element of Q has matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Now if y is the stabiliser of the point spanned by the second basis vector, the some element of G_y has matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

By Dickson's Theorem, $G \cap \mathrm{GL}(2, q)$ contains the group generated by these two matrices. This, combined with the fact that G is 2-transitive on the projective line, strongly restricts the possibilities. We find that either G contains $\mathrm{SL}(2, q)$, or G is a group of order 20 or 60 in $\Gamma\mathrm{L}(2, 4)$ (the latter being $\Gamma\mathrm{L}(1, 16)$). Note that the permutation groups induced on the projective line by these two semilinear groups are the same, isomorphic to the sharply 2-transitive Frobenius group of order 20.

6 Some geometry

Theorem 6.1 *Let G be a subgroup of* $\Gamma\mathrm{L}(n, q)$ *which is antiflag-transitive and is primitive on points. Then n is even and G is contained in* $\Gamma\mathrm{Sp}(n, q)$*. Moreover, either*

(a) G has rank 3 *on points, or*

(b) $n = 6$*, q is even, and* $G_2(q) \leq G \leq \mathrm{Aut}(G_2(q))$.

7 The inductive step for primitive groups

8 Imprimitive groups

9 Appendix: an orbit theorem

This appendix gives some orbit theorems for subgroups of $\Gamma\mathrm{L}(n, q)$, proved by using only simple facts about the permutation character. These theorems also follow from Kantor's theorem [5] and Block's Lemma, but the proofs here are more elementary. The theorems also have analogues for subgroups of the symmetric group S_n acting on subsets of $\{1,\ldots,n\}$.

According to the Orbit-counting Lemma, if G acts on Ω with permutation character π, then the number of orbits of G on Ω is equal to $\langle\pi,1\rangle_G$, where 1 denotes the principal character of G. If G also acts on Ω' with permutation character π', then the permutation character of G on $\Omega\times\Omega'$ is $\pi\pi'$, and so the number of orbits of G on $\Omega\times\Omega'$ is $\langle\pi\pi',1\rangle_G = \langle\pi,\pi'\rangle_G$. In particular, the rank of G on Ω is equal to $\langle\pi,\pi\rangle_G$.

Let V be an n-dimensional vector space over $\mathrm{GF}(q)$. For $0\le i\le n$, let P_i denote the set of i-dimensional subspaces of V, and let π_i denote the permutation character of $\Gamma\mathrm{L}(n,q)$ on P_i.

Lemma 9.1 *There are irreducible characters* $\chi_0,\chi_1,\ldots,\chi_{\lfloor n/2\rfloor}$ *of* $\Gamma\mathrm{L}(n,q)$ *such that*

$$\pi_i = \pi_{n-i} = \chi_0+\chi_i+\cdots+\chi_i$$

for $i\le n/2$.

Proof Let $G=\Gamma\mathrm{L}(n,q)$. We show first that $\pi_j=\chi_0+\chi_1+\cdots+\chi_j$ for $j\le n/2$. The proof is by induction on j, the result being clear for $j=0$ (since $|P_0|=1$).

We claim that, for $0\le i\le j\le n/2$, we have

$$\langle\pi_i,\pi_j\rangle_G = i+1.$$

Indeed, elementary linear algebra shows that, for $0\le k\le i$, the subset

$$\{(X,Y)\in P_i\times P_j : \dim(X\cap Y)=k\}$$

is an orbit; and these are all the orbits. Hence

$$\langle\pi_i-\pi_{i-1},\pi_j\rangle_G = 1.$$

By the inductive hypothesis, if $i<j$, then $\pi_i-\pi_{i-1}=\chi_i$; so χ_i occurs in π_j with multiplicity 1. We conclude that

$$\pi_j=\chi_0+\cdots+\chi_{j-1}+\psi$$

for some character ψ containing none of $\chi_0,\ldots,\chi_{j-1}$. The fact that $\langle\pi_j,\pi_j\rangle_G = j+1$ shows that ψ is irreducible. Taking $\chi_j=\psi$, we complete the inductive step.

Now again let $0\le i\le j\le n/2$. We claim that

$$\langle\pi_i,\pi_{n-j}\rangle_G = i+1 \text{ and } \langle\pi_{n-j},\pi_{n-j}\rangle_G = j+1.$$

This is proved by linear algebra as before: the orbits on $P_i \times P_{n-j}$ are

$$\{(X,Y) \in P_i \times P_{n-j} : \dim(X \cap Y) = k\}$$

for $k = 0, \ldots, i$, while the orbits on $P_{n-j} \times P_{n-j}$ are

$$\{(X,Y) \in P_{n-j} \times P_{n-j} : \dim(X \cap Y) = k\}$$

for $k = n-2j, \ldots, n-j$.

Then the same argument as before shows that π_{n-j} contains χ_i with multiplicity 1 for $i = 0, \ldots, j$, and nothing else. ■

We say that a character π of G is *contained in* a character π' if $\pi' = \pi + \psi$ for some character ψ. Now, if π and π' are permutation characters of G on Ω and Ω', and π is contained in π', then:

(a) $\langle \pi, 1 \rangle_G \leq \langle \pi', 1 \rangle_G$; that is, the number of orbits of G on Ω' is not less than the number of orbits on Ω;

(b) $\langle \pi, \pi \rangle_G \leq \langle \pi', \pi' \rangle_G$; that is, the rank of G on Ω' is not less than the rank on Ω;

Theorem 9.2 *Let G be any subgroup of $\Gamma\mathrm{L}(n,q)$, having N_i orbits on P_i for $0 \leq i \leq n$. Then the following hold:*

(a) $N_i = N_{n-i}$ for $0 \leq i \leq n/2$.

(b) $N_i \leq N_j$ for $0 \leq i \leq j \leq n/2$.

For the Lemma shows that the permutation characters of $\Gamma\mathrm{L}(n,q)$ on P_i and P_{n-i} are equal, and that the permutation character on P_i is contained in the character on P_j if $0 \leq i \leq j \leq n/2$; these facts remain true when the characters are restricted to the subgroup G. Obviously the analogous statement to (a) and (b) also holds if we replace “number of orbits” by “rank”. ■

10 Appendix: $G_2(q)$ and its hexagon

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