Enumerative Combinatorics 6: Symmetric polynomials

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A symmetric polynomial in n indeterminates is one which is unchanged under any permutation of the indeterminates. The theory of symmetric polynomials goes back to Newton, but more recently has been very closely connected with the representation theory of the symmetric group, which we glanced at in Lecture 3. I will just give a few simple results here. The best reference is Ian Macdonald's book *Symmetric Functions and Hall Polynomials*.

6.1 Symmetric polynomials

Let $x_1, \ldots, x_n$ be indeterminates. If π is a permutation of $\{1, \ldots, n\}$, we denote by $i\pi$ the image of i under π. Now a polynomial $F(x_1, \ldots, x_n)$ is a *symmetric polynomial* if

$$F(x_{1\pi}, \ldots, x_{n\pi}) = F(x_1, \ldots, x_n) \text{ for all } \pi \in S_n,$$

where S_n is the *symmetric group* of degree n (the group of all polynomials of degree n).

Any polynomial is a linear combination of monomials $x_1^{a_1} \cdots x_n^{a_n}$, where $a_1, \ldots, a_n$ are non-negative integers. The degree of this monomial is $a_1 + \cdots + a_n$. A polynomial is *homogeneous of degree* r if every monomial has degree r. Any polynomial can be written as a sum of homogeneous polynomials of degrees $1, 2, \ldots$.

In a homogeneous symmetric polynomial of degree r, the exponents in any monomial form a partition of r into at most n parts; two monomials which give rise to the same partition are equivalent under a permutation, and so must have the same coefficient. Thus, the dimension of the space of homogeneous symmetric polynomials of degree r is $p_n(r)$, the number of partitions of r with at most n parts.

There are three especially important symmetric polynomials:

(a) The *elementary symmetric polynomial* e_r, which is the sum of all the monomials consisting of products of r distinct indeterminates. Note that there are $\binom{n}{r}$ monomials in the sum.

(b) The *complete symmetric polynomial* h_r, which is the sum of all the monomials of degree r. There are $\binom{n+r-1}{r}$ terms in the sum: the proof of this is given in the Appendix to these notes.

(c) The *power sum polynomial* p_r, which is simply $\sum_{i=1}^{n} x_i^r$.

For example, if $n = 3$ and $r = 2$,

(a) the elementary symmetric polynomial is $x_1x_2 + x_2x_3 + x_1x_3$;

(b) the complete symmetric polynomial is $x_1x_2+x_2x_3+x_1x_3+x_1^2+x_2^2+x_3^2$;

(c) the power sum polynomial is $x_1^2 + x_2^2 + x_3^2$.

Note that $e_r(1,\ldots,n) = \binom{n}{r}$, $h_r(1,\ldots,1) = \binom{n+r-1}{r}$, and $p_r(1,\ldots,1) = n$.

Also, the q-binomial theorem that we met in the last lecture shows that

$$e_r(1, q, q^2, \ldots, q^{n-1}) = q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix}_q ,$$

and Heine's formula shows that, similarly,

$$h_r(1, q, q^2, \ldots, q^{n-1}) = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_q .$$

6.2 Generating functions

The best-known occurrence of the elementary symmetric polynomials is the connection with the roots of polynomials. (To avoid conflict with x_i, the variable in a polynomial is t in this section.) The coefficient of t^{n-r} in a polynomial of degree n is $(-1)^r e_r(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are the roots. This is because the polynomial can be written as

$$(t - a_1)(t - a_2) \cdots (t - a_n),$$

and the term in t^{n-r} is formed by choosing t from $n - r$ of the factors and $-a_i$ from the remaining r.

Said otherwise, and putting $x_i = -1/a_i$, this says that the generating function for the elementary symmetric polynomials is

$$E(t) = \sum_{r=0}^{n} e_r(x_1, \ldots, x_n) t^r = \prod_{i=1}^{n} (1 + x_i t),$$

with the convention that $e_0 = 1$.

In a similar way, the generating function for the complete symmetric polynomials is

$$H(t) = \sum_{r \geq 0} h_r(x_1, \ldots, x_n) t^r = \prod_{i=1}^{n} (1 - x_i t)^{-1}.$$

We also take $P(t)$ to be the generating function for the power sum polynomials, with a shift:

$$P(t) = \sum_{r \geq 1} p_r(x_1, \ldots, x_n) t^{r-1}.$$

Now we see that $H(t) = E(-t)^{-1}$, so that

$$\sum_{r=0}^{n} (-1)^r 3_r h_{n-r} = 0 \text{ for } n \geq 1.$$

For $P(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = P(t) H(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t} E(t) = P(-t) E(t).$$

6.3 Functions indexed by partitions

We extend the definitions of symmetric polynomials as follows. Let $\lambda = (a_1, a_2, \ldots)$ be a partition of r, a non-decreasing sequence of integers with sum r. Then, if z denotes one of the symbols e, h or p, we define z_λ to be the product of z_{a_i} over all the parts a_i of λ; this is again a symmetric polynomial of degree r. For example, if $n = 3$ and λ is the partition $(2,1)$ of 3, we have

$$\begin{aligned} e_\lambda &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3), \\ p_\lambda &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3), \\ h_\lambda &= e_\lambda + p_\lambda. \end{aligned}$$

We also define the *basic polynomial* m_λ to be the sum of all monomials with exponents $a_1, a_2, \ldots$. In the above case,

$$m_\lambda = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

Theorem 6.1 *If $n \geq r$, and z is one of the symbols m, e, h, p, then any symmetric polynomial of degree r can be written uniquely as a linear combination of the polynomials z_λ, as λ runs over all partitions. Moreover, in all cases except $z = p$, if the polynomial has integer coefficients, then it is a linear combination with integer coefficients.*

So the polynomials e_r or h_r, with $r \leq n$, are generators of the ring of symmetric polynomials in n variables with integer coefficients. For $z = e$, this is a version of Newton's Theorem on symmetric polynomials (which, however, applies also to rational functions).

6.4 Appendix: Selections with repetition

Theorem 6.2 *The number of n-tuples of non-negative integers with sum r is $\binom{n+r-1}{r}$.*

The claim about the number of monomials of degree r follows immediately from this result, which should be contrasted with the fact that the number of n-tuples of zeros and ones with sum r is $\binom{n}{r}$.

Proof We can describe any such n-tuple in the following way. Take a line of $n + r - 1$ boxes. Then choose $n - 1$ boxes, and place barriers in these boxes. Let

(a) a_1 be the number of empty boxes before the first barrier;

(b) a_2 be the number of empty boxes between the first and second barriers;

(c) ...

(d) a_n be the number of empty boxes after the last barrier.

Then $a_1, \dots, a_n$ are non-negative integers with sum r. Conversely, given n non-negative integers with sum r, we can represent it with $n - 1$ barriers in $n+r-1$ boxes: place the first barrier after a_1 empty boxes, the second after a_2 further empty boxes, and so on.

So the required number of n-tuples is equal to the number of ways to position $n - 1$ barriers in $n + r - 1$ boxes, which is

$$\binom{n+r-1}{n-1} = \binom{n+r-1}{r},$$

as required.