Enumerative Combinatorics 4: Unimodality

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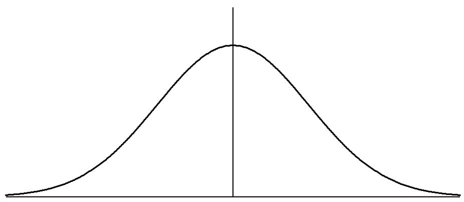
It is well known that the binomial coefficients increase up to halfway, and then decrease. Indeed, the shape of the bar graph of binomial coefficients is well approximated by a scaled version of the "bell curve" of the normal distribution.



This property of binomial coefficients is easily proved. Since

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

the binomial coefficient increases from k to $k+1$, remains constant, or decreases, according as $n-k > k+1$, $n-k = k+1$ or $n-k < k+1$, that is, as n is greater than, equal to, or less than $2k+1$. So, if n is even, the binomial coefficients increase up to $k = n/2$ and then decrease; if n is odd, the two middle values ($k = (n-1)/2$ and $k = (n+1)/2$) are equal, and they increase before this point and decrease after.

Other combinatorial numbers also show this unimodality property, but in cases where we don't have a formula, we need general techniques.

4.1 Unimodality and log-concavity

Given a sequence of positive numbers, say $a_0, a_1, a_2, \ldots, a_n$, we say that the sequence is *unimodal* if there is an index m with $0 \leq m \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

The sequence $a_0, a_1, a_2, \ldots, a_n$ of positive integers is said to be *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for $1 \leq k \leq n-1$. The reason for the name is that the logarithms of the as are concave: setting $b_k = \log a_k$, we have $2b_k \leq b_{k-1} + b_{k+1}$, or in other words, $b_{k+1} - b_k \leq b_k - b_{k-1}$. So if we plot the points (k, b_k) for $0 \leq k \leq n$, then the slopes of the lines joining consecutive points decrease as k increases, so that the figure they form is concave when viewed from above.

Now it is clear that a log-concave sequence is unimodal.

A nice general result is:

Theorem 4.1 *Let* $A(x) = \sum_{k=0}^{n} a_k x^k$ *be the generating polynomial for the numbers* $a_0, \ldots, a_n$*. Suppose that all the roots of the equation* $A(x) = 0$ *are real and negative. Then the sequence* $a_0, \ldots a_n$ *is log-concave.*

Before we begin the proof, we note that a polynomial with all coefficients positive cannot have a real non-negative root, and a polynomial all of whose roots are negative has all its coefficients positive.

The proof is by induction: there is nothing to prove when $n = 1$, since any sequence of two numbers is log-concave. For $n = 2$, the condition for the polynomial $a_0 + a_1x + a_2x^2$ to have real roots is $a_1^2 - 4a_0a_2 \geq 0$, which is stronger than log-concavity; as remarked, if the roots are real, they are negative.

Now we turn to the general case. Suppose that $A(x) = (x+c)B(x)$, where $c > 0$ and

$$B(x) = b_{n-1}x^{n-1} + \cdots + b_1x + b_0.$$

Now the polynomial $B(x)$ has all its roots real and negative, since they are all the roots of $A(x)$ except for $-c$. So the coefficients are all positive, and by the inductive hypothesis, the sequence $b_0, \ldots, b_{n-1}$ is log-concave; that is,

$$b_k^2 \geq b_{k-1}b_{k+1}$$

for $k = 1, \ldots, n-2$. Also, since $A(x) = (x + c)B(x)$, we have $a_0 = cb_0$, $a_n = b_{n-1}$, and $a_k = b_{k-1} + cb_k$ for $1 \le k \le n-1$.

We first show that $b_k b_{k-1} \ge b_{k+1} b_{k-2}$ for $2 \le k \le n-2$. For we have

$$b_k^2 b_{k-1} \ge b_{k+1} b_{k-1}^2 \ge b_{k+1} b_k b_{k-2};$$

dividing by b_k gives the result.

Now for $2 \le k \le n-2$, we have

$$\begin{aligned} a_k^2 - a_{k+1}a_{k-1} &= (b_{k-1} + cb_k)^2 - (b_k + cb_{k+1})(b_{k-2} + cb_{k-1}) \\ &= (b_{k-1}^2 - b_k b_{k-2}) + c(b_{k-1}b_k - b_{k+1}b_{k-2}) + c^2(b_k^2 - b_{k+1}b_{k-1}); \end{aligned}$$

and all three terms are non-negative since $c > 0$.

The cases $k = 1$ and $k = n - 1$ are left to the reader.

4.2 Binomial coefficients and Stirling numbers

For the binomial coefficients, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n;$$

all its roots are -1, and so the theorem shows that the binomial coefficients are log-concave, and hence unimodal.

For the unsigned Stirling numbers of the first kind, we have

$$\sum_{k=1}^{n} u(n,k)x^k = x(x + 1) \cdots (x + n - 1),$$

and the polynomial on the right has roots $0, -1, -2, \ldots, -(n-1)$. We can neglect the zero root: the Stirling numbers start at $k = 1$ rather than zero, and dividing by x simply changes the indexing so that they start at 0. So again the Stirling numbers are log-concave and hence unimodal.

The Stirling numbers of the second kind are more difficult, since there is no convenient form for the generating polynomial. We start with the recurrence relation

$$S(n, 1) = S(n, n) = 1, \quad S(n, k) = S(n-1, k-1) + kS(n-1, k) \text{ for } 1 < k < n.$$

Let

$$A_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

We have $A_0(x) = 1$. For $n > 0$, we have $A(n,0) = 0$, so zero is a root of $A_n(x) = 0$. We have to show that the other roots are real and negative. We prove this by induction: $P_1(x) = x$ has a single root at $x = 0$, while $A_2(x) = x + x^2$ has roots at $x = 0$ and $x = -1$; so the induction begins.

From the recurrence relation, we have

$$\begin{aligned} A_n(x) &= \sum_{k=1}^{n} S(n,k)x^k \\ &= \sum_{k=1}^{n} S(n-1,k-1)x^k + \sum_{k=1}^{n} kS(n-1,k)x^k \\ &= x\left(\mathrm{d}A_{n-1}(x)/\mathrm{d}x + A_{n-1}(x)\right). \end{aligned}$$

Putting $B_n(x) = A_n(x)\mathrm{e}^x$, we see that $A_n(x) = 0$ and $B_n(x) = 0$ have the same roots. The identity above, multiplied by e^x, gives

$$x\,\mathrm{d}B_{n-1}(x)/\mathrm{d}x = B_n(x).$$

By Rolle's Theorem, there is a root of $B_n(x)$ between each pair of roots of $B_{n-1}(x)$, and one to the left of the smallest root of $B_{n-1}(x)$ (since $B_{n-1}(x) \to 0$ as $x \to -\infty$); and also a a root at 0. This accounts for $(n-2)+1+1$ roots, that is, all the roots of $B_n(x)$. So the induction step is complete.

Exercises

1 Let S be a fixed set of positive integers, and let r_n be the number of partitions of n into distinct parts from the set S. What is the generating polynomial $\sum r_n x^n$? Is the sequence (r_n) unimodal?

2 Let (a_n) be an infinite sequence of positive numbers which is log-concave (that is, $a_{n-1}a_{n+1} \leq a_n^2$ for all $n \geq 1$). Show that the ratio a_{n+1}/a_n tends to a limit as $n \to \infty$.