A problem held over from last week

1. Let $U$ be a subspace of the space of $m \times n$ complex matrices which contains no matrix of rank 1. Prove that \( \dim(U) \leq (m-1)(n-1) \).

   Is this true for real matrices?

   Hence [or assuming the result if you can’t prove it] show that if

   $$ A = \bigoplus_{i \geq 0} V_i $$

   is a graded algebra over \( \mathbb{C} \) (that is, \( V_0 = \mathbb{C} \cdot 1 \) and \( V_i \cdot V_j \subseteq V_{i+j} \)) which is an integral domain (i.e. has no divisors of zero), then

   $$ \dim(V_{i+j}) \geq \dim(V_i) + \dim(V_j) - 1. $$

   Deduce that, if $G$ is an oligomorphic permutation group for which the graded algebra $A^G$ is an integral domain, then

   $$ f_{i+j}(G) \geq f_i(G) + f_j(G) - 1. $$

   Find an example where equality holds for all $i$ and $j$.

Bases and Schreier–Sims algorithm

2. Show that the permutations $$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$ and $$(1,2,3)(4,5,7)(8,9,11)$$ generate a 2-transitive permutation group of degree 12. (Actually the group generated by these two permutations is the Mathieu group $M_{12}$, which is sharply 5-transitive: if you enjoy this kind of calculation, you may want to prove this.)

3. Let $G$ be the group induced on 2-sets by the symmetric group on \( \{1,2,\ldots,m\} \). Find the minimum base size and the size(s) of bases produced by the greedy algorithm for $G$.

4. Prove that if the permutation group $G$ has irredundant bases of sizes $m_1$ and $m_2$, then it has irredundant bases of all possible sizes between $m_1$ and $m_2$.

   Give an example to show that this is false for minimal bases.
Matroids

5. The uniform matroid $U(k,n)$ is defined as follows: $E = \{1,2,\ldots,n\}$, and the independent sets are all subsets of cardinality at most $k$.

- What are the bases, the circuits, and the hyperplanes, of $U(k,n)$?

- Show that if $G$ is a permutation group on $E$ with the property that the stabiliser of any $k$ points is the identity but any $k-1$ points are fixed by some non-identity element, show that the irredundant bases for $G$ are the bases of $U(k,n)$.

- [Harder!] Show that a group with the above property is $(k-1)$-transitive if $k > 1$. Hint: Use induction on $k$. The most difficult case is the starting case $k = 2$.

Notes: (a) Prove that, if a group with this property with $k = 2$ has degree $n$ and is transitive, then it contains exactly $n-1$ derangements. Frobenius' Theorem asserts that these derangements, together with the identity, form a normal subgroup of $G$ which acts regularly; you may use this if you wish, but it is possible to do without.

(b) All groups satisfying this property with $k \geq 3$ have been determined; it is not necessary to use the Classification of Finite Simple Groups in this classification. In particular, if $k \geq 5$, then such a group must be sharply $k$-transitive.