Problems from the DocCourse: Day 7

Cycle index and orbit counting

Recall that, if $G$ is an oligomorphic permutation group, then $\hat{Z}(G)$ is the sum of the cycle indices of $G(A)$, where $A$ runs over a set of orbit representatives on finite sets and $G(A)$ is the group of permutations of $A$ induced by its setwise stabiliser. Also $F_G(x)$ is the exponential generating function for the number of orbits of $G$ on $n$-tuples of distinct points, and $f_G(x)$ the ordinary generating function for the number of orbits of $G$ on $n$-sets. They are obtained from $\hat{Z}(G)$ by the specialisations $s_1 \leftarrow x$, $s_i \leftarrow 0$ for $i > 1$, resp. $s_i \leftarrow x^i$ for all $i$.

$F_G^*(x)$ is the exponential generating function for the number of orbits of $G$ on all $n$-tuples (repeats allowed).

1. Let $B$ be the group of permutations which preserve or reverse the order of the rational numbers. Calculate $\hat{Z}(B)$, and hence evaluate $F_B(x)$ and $f_B(x)$. Compare these with what you expect.

Is it possible to assign a value to $F_B(-1)$?

2. Same question for $D$, the group of permutations which preserve or reverse the circular order on the set of complex roots of unity.

3. Let $G$ be the permutation group induced by the symmetric group $S_n$ acting on the set of all 2-element subsets of $\{1, \ldots, n\}$.

   - Show that the specialisation $s_i \leftarrow 1 + x^i$ gives a polynomial $p$ in which the coefficient of $x^m$ is the number of graphs on $n$ vertices and $m$ edges, up to isomorphism.

   - Let $g$ be an element of $S_n$ having $c_i$ cycles of length $i$ (in the usual action). Show that each cycle of length $i$ contributes $(i - 1)/2$ cycles of length $i$ (if $i$ is odd) or $(i - 2)/2$ cycles of length $i$ and one of length $i/2$ (if $i$ is even) in the action on 2-sets. Show also that each pair consisting of cycles of lengths $i$ and $j$ contributes $\gcd(i, j)$ cycles of length $\text{lcm}(i, j)$.

   - Hence calculate the cycle index of $S_4$ on 2-sets, and enumerate the 4-vertex graphs by number of edges. Check by listing the graphs.
4. Show that the limit of the proportion of derangements in the group $S_n$ acting on 2-sets, as $n \to \infty$, is $2e^{-3/2}$.

Is it possible to assign a value to $F_G(-1)$, where $G$ is the infinite symmetric group in its action on 2-sets? [There is no answer to this question!]

Calculate the limiting proportion of derangements of $S_n$ acting on 3-sets as $n \to \infty$.

5. Let $G = C_2 \text{Wr} A$, where $C_2$ is the cyclic group of order 2 acting regularly, and $A$ is the group of order-preserving permutations of $\mathbb{Q}$. Show that $f_n(G)$ is the $n$th Fibonacci number.

6. Let $F_n^*(G)$ denote the number of orbits of $G$ on all $n$-tuples of points of $\Omega$ (repetitions allowed). Prove that

$$F_n^*(G) = F_n(G \text{Wr} S).$$

7. For $i = 1, 2$, let $G_i$ be an oligomorphic permutation group on $\Omega_i$. Let $G = G_1 \times G_2$ acting coordinatewise on $\Omega_1 \times \Omega_2$, that is, $(\omega_1, \omega_2)(g_1, g_2) = (\omega_1 g_1, \omega_2 g_2)$.

Prove that

$$F_n^*(G_1 \times G_2) = F_n^*(G_1)F_n^*(G_2).$$

Prove that, with this action, $F_n(A \times A)/n!$ is equal to the number of matrices with entries 0 and 1 which have exactly $n$ ones and have no row or column consisting entirely of zeros.

Hence find a formula for this number.