Problems from the DocCourse: Day 5

Two problems from Antonio Machì

1. A descent in a permutation $g$ in the symmetric group $S_n$ (on the set $\{1,2,\ldots,n\}$) is a point $i$ such that $ig \leq i$; it is a strict descent if $ig < i$.

Prove that, if a subgroup $G$ of $S_n$ has $h$ orbits, then the average number of descents of a permutation in $G$ is $(n + h)/2$, and the average number of strict descents is $(n - h)/2$. Deduce the Orbit-Counting Lemma.

2. A combinatorial proof of Hurwitz’s Theorem. You don’t need to know anything about maps or Riemann surfaces!

(a) Let $z(g)$ be the number of cycles of a permutation $g \in S_n$, and $t(g)$ the minimum number of transpositions whose product is $g$. Prove that

$$z(g) + t(g) = n.$$

(b) Prove that, if $t_1, \ldots, t_k$ are transpositions which generate a transitive subgroup of $S_n$, then $k \geq n - 1$. If, further, $t_1 \cdots t_k = 1$, then $k \geq 2n - 2$ and $k$ is even. [Hint: Think of the $t_i$ as edges of a graph.]

(c) Hence show that, if $g_1, \ldots, g_m$ generate a transitive subgroup of $S_n$, then

$$z(g_1) + \cdots + z(g_m) \leq (m - 1)n + 1.$$

If, further, $g_1 \cdots g_m = 1$, then

$$z(g_1) + \cdots + z(g_m) \leq (m - 2)n + 2,$$

and the difference of these two quantities is even.

(d) How should the preceding result be modified if the group generated by $g_1, \ldots, g_m$ has a prescribed number $p$ of orbits?

(e) Suppose that $g_1, g_2, g_3$ generate a regular subgroup $G$ of $S_n$, and $g_1g_2g_3 = 1$. Let $z(g_1) + z(g_2) + z(g_3) = n + 2 - 2g$. Prove Hurwitz’s Theorem:

If $g \geq 1$, then the order of $G$ is at most $84(g - 1)$.

Construct an example meeting the bound when $g = 3$. [Hint: If $|G| = n$ and $g_i$ has order $n_i$ for $i = 1, 2, 3$, show that

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 - \frac{2(g - 1)}{n}.$$]
Problems on homogeneous structures

1. Let $G_n$ be the class of finite graphs containing no complete subgraph on $n$ vertices. Prove that $G_n$ has the amalgamation property. Let $H_n$ be the Fraïssé limit of this class, and $G_n = \text{Aut}(H_n)$. (The graphs $H_n$ were first constructed by Henson.)

Prove that, if $n = 3$, then the stabiliser of a vertex $v$ in $G_n$ acts highly transitively on the set of neighbours of $v$, but contains no finitary permutation.

Prove that, if $n > 3$, then the stabiliser of a vertex $v$ in $G_n$, acting on the set of neighbours of $v$, is isomorphic to a subgroup of $G_{n-1}$.

2. Prove that the class of finite bipartite graphs does not have the amalgamation property.

Let $B$ be the class of finite bipartite graphs with a distinguished bipartite block. Show that $B$ has the amalgamation property. Let $B$ be its Fraïssé limit, and $G = \text{Aut}(B)$. Prove that $G$ has two orbits on the set of vertices of $B$, and is highly transitive on each orbit but contains no finitary permutation.

3. This exercise is due to Sam Tarzi.

Let $L$ be the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$. (If you know about lattices, do this question for an arbitrary lattice in $\mathbb{R}^d$.)

Given a finite set $S$ of points of $L$, and a positive real number $r$, prove that there is a point $v \in L$ such that the Euclidean distances $||v - x||$, for $x \in S$, are all different and all greater than $r$.

Now let $(d_1, d_2, \ldots)$ be the list of all distances between pairs of points of $L$. Define a graph on the vertex set $L$ by deciding independently at random, for each $i$, whether all pairs of points at distance $d_i$ are edges or all are non-edges. Show that, with probability 1, this graph is isomorphic to the countable random graph $R$.

Deduce that the isometry group of $L$ is a subgroup of $\text{Aut}(R)$.

4∗∗. A boron tree is a finite tree in which all vertices have degree 1 or 3. Let $X$ be the class of finite relational structures with a quaternary relation (written $(ab|cd)$) defined as follows: the points of the structure are the leaves of a boron tree; the relation $(ab|cd)$ holds if and only if $a, b, c, d$ are all distinct and the paths joining them form a tree homeomorphic to the following:

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    a
     \\
  b  |  c
     \\
  d
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Prove that $X$ has the amalgamation property. If $X$ is its Fraïssé limit, and $G = \text{Aut}(X)$, prove that $G$ is 3-transitive but not 4-transitive, and is 5-set-transitive but not 6-set-transitive.