Problems from the DocCourse: Day 3

Problems

1. (a) Prove that, if there exists a finite transitive permutation group which contains no derangement of prime power order, then there exists a primitive simple permutation group with this property.

   [Hint: If $G$ has the property and has a nontrivial congruence, then the permutation group induced by $G$ on the set of congruence classes has the property. Also, if $G$ has the property, then so does any transitive subgroup.]

   (b) Show that if $G$ is a primitive permutation group containing no derangement of prime power order, then the stabiliser of a point is a maximal proper subgroup of $G$ which intersects every conjugacy class of elements of prime power order in $G$.

   (c) Prove that $A_5$ (the alternating group of degree 5) does not contain a maximal subgroup which meets every conjugacy class of elements of prime power order.

2. (a) Let $G$ be a transitive permutation group of degree $n = p^a \cdot b$ where $a$ does not divide $b$. Prove that the orbits of a Sylow $p$-subgroup of $G$ have size at least $p^a$.

   (b) Prove that, if $P$ is a transitive $p$-group, then more than $(p - 1)|P|/p$ elements of $P$ are derangements. [Hint: Show that $P$ has an intransitive subgroup of index $p$.] Deduce that a $p$-group with fewer than $p$ orbits contains a derangement.

   (c) Hence show that if $G$ is a transitive permutation group of degree $n = p^a \cdot b$ where $a > 0$ and $b < p$, then $G$ contains a derangement of order a power of $p$.

From the book

1.31**, 1.33*, 4.3*, 4.7*, 4.22**.