Problems from the DocCourse: Day 2

Project: Factorising the cycle index

If the permutation group G is a direct product $G = G_1 \times G_2$, acting on the disjoint union of the sets on which G_1 and G_2 act, as usual, then

$$Z(G) = Z(G_1) \cdot Z(G_2).$$

Apart from this, it seems very rare for the cycle index to factorise (as a polynomial over the integers). Indeed, I know only one example of a *transitive* group G for which $Z(G)$ factorises.

1. Show that, if $Z(G) = F \cdot F'$, then all the monomials in F have the same weight (where the *weight* of a monomial $s_1^{b_1} s_2^{b_2} \cdots s_n^{b_n}$ is defined to be $b_1 + 2b_2 + \cdots + nb_n$. (Note that every term in $Z(G)$ has weight equal to the degree of G.)

2. Find any examples that you can of groups G for which the cycle index has a factorisation which is not "explained" by G being a direct product.

3. Prove that the cycle index of the symmetric or alternating group is irreducible.

4. Prove that if the cycle index of the transitive group G involves only s_1 and s_2, then it is irreducible. (I think this is true but have not checked all details.) (Hint: What is the structure of such a group?)

5. If you can, determine all transitive groups whose cycle index is reducible. Failing this, prove further results like 3 and 4 above for other classes of transitive groups.

Standard problems

From the book: 1.10^*, 1.11^*, 2.4^*, 2.5^*, 2.14^{**}.