Donald at Queen Mary: Climbing walls and PLRs

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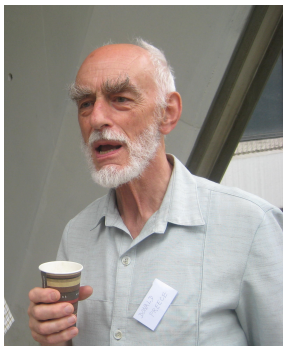
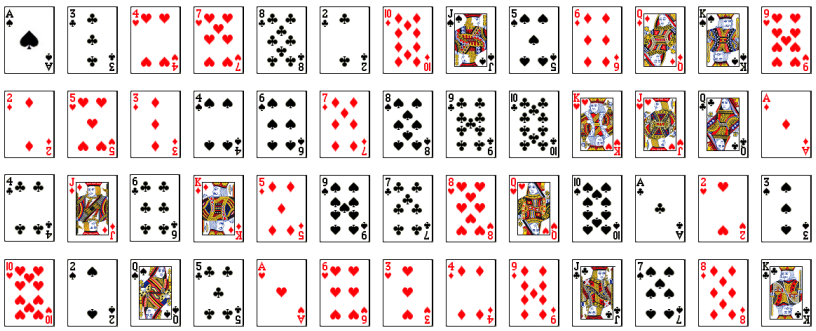
Donald Preece Memorial Day, 17 September 2015





Our first meeting

I first met Donald at the BCC in Aberystwyth in 1973, which I think was his first BCC, perhaps his first combinatorics conference. Rosemary has told this story. After his talk, I sat next to him on the excursion coach, and the result of that discussion was a joint publication constructing some designs resembling the one on the title page of these slides.

This is a table from the paper (mentioned by Rosemary in her talk). I didn't understand the exact relation between Donald's and my points of view for more than 20 years, when I found an infinite family of these designs.

AAA BBB CCC DDD EEE FFF GGG HHH III JJJ KKK LLL MMM NNN OOO PPP
HGB GHA FED EFC DCF CDE BAH ABG FOJ OFI NML MNK LEN KLM JIP IJO
KJD LIC ILB JKA OHH PMG MPF MOE CBL DAK ADJ BCI GFP HEO EIN PGM
MIE NJF OKG PLH IMA JNB KOC LPD EAM FBN GCO NDP AEI BFJ CGK DHL
OFL PEK MNJ NGI KBP LAO IDN JCM GND HMC EPB FOA GJH DIG ALF BKE
PCN ODM NAP MBO LGJ KHI JEL IPK HKF GLE FIH EJG DOB CPA BMD ANC
BDC ACD DBA CAB PNG KGH HPE GEF JLK IKL LJI KIJ NFO MOP PNM GMN
CKI DLJ AIK BJL GOM HPM EMO FNP ECA LDB IAC JBD OGE PHF MEG NPH
DMP CNO BON APM HIL GJK FCJ ELI LEH KPG JGF IHE PAD OBC NCB MDA
EBF FAE GDH HCG AFB BEA CND DOC MJN NIM OLP FKO INJ JMI EPL LOK
FLO EKP ILM GIN BPK ADL DNI CMJ NDG MGH PBE OAF JNC IGD LFA KEB
GPJ HOI ENL FMC CLN DHM AJP BIO OHB PGA MFD NKC KBF LCE ISH JAG
IEM JFH KGO LHP MAI NBJ OCK PBL AME BVF COG DPH KIA FJB GDC HLD
JOH IPG LMF KNE NKD MLC PIB OJA BGP AHO DEN CFM FCL EMK HAJ GBI
LNG KMH JPE IOF PJC OID NLA MKB DPO CEP BHM AGN HBK GAL FDI ECJ
NBK MGL PFI OEJ JDO ICP LBM KAN FPC EOB HNA GMB BLG AKH DJE CIF

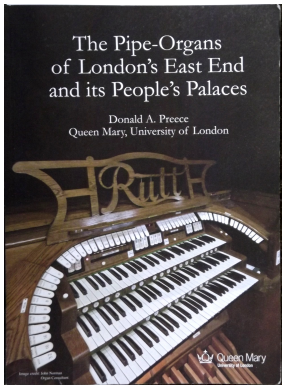
Donald and the BCC

With the BCC chair's hat on, I would like to say a little about Donald's contribution to the British Combinatorial Conferences.
Many BCC delegates over the years will know Donald's organisation of the conference concert. The amount of energy he put into this, both physical and nervous, was phenomenal. He would act as accompanist when required for almost anything.
But his biggest contribution occurred in 1999. The committee found itself without a conference venue, due to circumstances beyond our control. Donald stepped in and, with John Lamb's help, organised a very successful BCC at the University of Kent at Canterbury.

Donald at Queen Mary

Donald came to Queen Mary as a research professor in 2000. I immediately had to revise the impression of the very formal gentleman that he had made on me in Aberystwyth. He signed up for courses at the climbing wall in Mile End Park with great enthusiasm, and was soon abseiling down tall buildings for charity and scaling industrial chimneys all over the country. He was on the point of starting sky-diving until a detached retina put paid to this.
He also became involved with the Luncheon Club at Queen Mary, and through this, became involved with the Organ in the Great Hall, which was then in very poor condition. He was very much concerned with the refurbishment of the organ, and one of his compositions was played at its re-inauguration in 2013.

This is the cover of Donald's remarkable survey of East End organs, published by QMUL in 2012.
The cover picture shows the console of the refurbished organ, which we will hear later this afternoon.
His two copies of the book are both heavily annotated ...



Terraces, daisy chains, tredoku and more

Donald threw himself into research at Queen Mary. He studied various kinds of neighbour-balanced designs with Ian Anderson, Matt Ollis, and others. He returned to tight single-change covering designs. He constructed some new types of Youden rectangles which he had been seeking for many years.

He was always keen to stand in for a colleague and give a lecture to undergraduates. The Combinatorics Study Group also saw a number of his inimitable performances. Typical titles were "Daisy chains" on 16 March 2007, and "If at first you don't succeed ... a combinatorial breakthrough" on 22 October 2010.

At my retirement conference in 2013, he posed various challenges concerned with tredoku, a 3-dimensional version of Sudoku which appeared in *The Times*. Afterwards, quite a few members of the audience could be seen trying their hand at these.

Primitive lambda-roots

I want to discuss two related pieces of work I did with Donald during his time at Queen Mary, University of London, on primitive lambda-roots and on generators in arithmetic progression.

A primitive root modulo an integer n is an integer r which is coprime to n and has the property that every integer coprime to n is congruent to a power of r. For example, 3 is a primitive root mod 5, since $3^1 \equiv 3$, $3^2 \equiv 4$, $3^3 \equiv 2$, and $3^4 \equiv 1$.

Primitive roots do not exist for every integer: only numbers which are an odd prime power, twice an odd prime power, or 4 have them.

It is well known to those in Donald's field that primitive roots modulo a prime, or more generally in a finite field, are useful in various combinatorial constructions. But what are we to do if we need a design where the number of points is not prime (a frequent occurrence in statistics)?

I will give one of Donald's constructions which shows how he ingeniously bridged the gap. The next slide is in Donald's words.

Consider the following sequence of the elements of $\mathbb{Z}_{35}$:

START
10 15 5 3 9 27 11 33 29 17 16 13 4 12 1 21 7 ↘
0
25 20 30 32 26 8 24 2 6 18 19 22 31 23 34 14 28 ↙
FINISH

The last 17 entries, in reverse order, are the negatives of the first 17, which, with the zero, can also be written

$$5^5 \quad 5^6 \quad 5^7 \mid 3^1 \quad 3^2 \quad 3^3 \quad 3^4 \quad 3^5 \quad 3^6 \quad 3^7 \quad 3^8 \quad 3^9 \quad 3^{10} \quad 3^{11} \quad 3^{12} \mid 7^4 \quad 7^5 \mid 0.$$

If we write the respective entries here as x_i ($i = 1, 2, \ldots, 18$), then the successive differences $x_{i+1} - x_i$ ($i = 1, 2, \ldots, 17$) are

$$5 \quad -10 \quad -2 \quad 6 \quad -17 \quad -16 \quad -13 \quad -4 \quad -12 \quad -1 \quad -3 \quad -9 \quad 8 \quad -11 \quad -15 \quad -14 \quad -7.$$

Ignoring minus signs, these differences consist of each of the values $1, 2, \ldots, 17$ exactly once. This is a special type of terrace.

Carmichael's lambda-function $\lambda(n)$ is the maximum order of an element in the group of units of $\mathbb{Z}_n$, the integers mod n. (That is, the largest number of distinct powers we can get modulo n from a fixed element coprime to n.) An element of the group of units U_n is a primitive lambda-root if its order is $\lambda(n)$. Thus, if n is prime, $\lambda(n) = n - 1$ and primitive lambda-roots are just primitive roots.

In the preceding example, $\lambda(35)$ is the least common multiple of $\lambda(5) = 4$ and $\lambda(7) = 6$, that is, $\lambda(35) = 12$. Now 3 is a primitive lambda-root mod 35: its powers mod 35 are

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 11, \quad 3^5 = 33, \quad 3^6 = 29,$$
$$3^7 = 17, \quad 3^8 = 16, \quad 3^9 = 13, \quad 3^{10} = 4, \quad 3^{11} = 12, \quad 3^{12} = 1.$$

Motivated by this, Donald and I embarked on a study of primitive lambda-roots. We never found a suitable place to publish it, but you can access the notes (and the GAP functions I wrote for computing with them) at
`https://cameroncounts.wordpress.com/lecture-notes/`
(I should add that I never persuaded Donald to use the computer to do these calculations: he worked on paper on the train journey to London from East Malling, and presented me with his findings and his challenges, when he arrived.)

The notes are mainly expository, and contain many open problems. There are some unexpected connections. For example, if $\lambda^*(m)$ is the greatest n such that $\lambda(n) = m$, then $\lambda^*(2m)$ is also the denominator of the Bernoulli number B_{2m}, re-scaled. We give a proof, but I don't really understand why. (In fact, we found the key in a paper on mathematical physics!) It was also characteristic of Donald that he invented names for PLRs having some special property in which he was interested. I assume that these were properties which had proved useful in his constructions, but I never found out more. Thus a PLR could be negating or non-negating, inward or outward, perfect, imperfect or aberrant.

Generators in arithmetic progression

This investigation led us on to further exploration of the group U_n of units in $\mathbb{Z}_n$. I guess that Donald had some combinatorial constructions in mind, but I have no idea what they were. As with so many things he did, the work was driven by examples. Here are two. We write

$$U_n = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c$$

to denote that U_n is the direct product of cyclic subgroups generated by x, y, z, and that the orders of these elements are a, b, c respectively.

$$U_{61} = \langle 9 \rangle_5 \times \langle 11 \rangle_4 \times \langle 13 \rangle_3,$$

where the orders as well as the generators themselves are in arithmetic progression; and

$$U_{455} = \langle 92 \rangle_4 \times \langle 93 \rangle_{12} \times \langle 94 \rangle_6,$$

where the generators are consecutive and the orders are even.

The way we worked was that Donald would arrive at Queen Mary with a new "theorem", based on his extensive hand calculations, and it was my job to write down a proof of the theorem.
I didn't always succeed, and there are *many* open problems in the paper. Here is one case where I did. But even this raises number-theoretic questions such as whether an infinity of such primes exists. (Donald produced long lists by hand.)

Theorem

Let n be a prime congruent to 7 or 31 (mod 36), $n > 7$. Suppose that the roots x_1 and x_2 of $x^2 + 3x + 3 = 0$ in $\mathbb{Z}_n$ have orders $(n-1)/2$ and $n-1$ respectively. Then

$$U_n = \langle 2x_2 + 3 \rangle_m \times \langle x_2 + 1 \rangle_3 \times \langle -1 \rangle_2,$$

where $m = (n-1)/6$.

This and two similar theorems covered all cases of three generators in AP with orders 2, 3 and $(n-1)/6$ when n is prime.

Among the other things we did in the paper were:

- A "lifting" technique that enabled us to use results about primes to study composite n.
- Some examples (but not much theory) about the analogous problem in finite fields (we gave examples in fields of orders 11^2, 11^3, 19^2, 19^3, 23^2 and 29^2).
- A couple of isolated examples of 4-term arithmetic progressions of generators: for example,

$$U_{104} = \langle 77 \rangle_2 \times \langle 79 \rangle_2 \times \langle 81 \rangle_3 \times \langle 83 \rangle_4.$$

We remarked that we had been unable to find decompositions with more than four terms; this is an open problem.

Donald's legacy

Donald left a large number of pieces of paper and computer files. He also left indelible memories, some of which we are sharing today.
But he also left us a mathematical legacy of ideas which are not yet completely worked out or published. He wrote to co-authors in an email in 2010,

> I'd better not say my *Nunc Dimittis* until I've written it up properly! (If I don't survive that long, any of you should feel free to complete the task.)

If anyone would like to help me with my part of this task, I would welcome your assistance!