Roots $x_k(y)$ of a formal power series

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y) x^n$$

with applications to graph enumeration
and $q$-series

Alan Sokal
New York University / University College London

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LECTURE #3

The leading root of the partial theta function
The basic set-up, reviewed

• Start from a formal power series

\[ f(x, y) = \sum_{n=0}^{\infty} a_n(y) x^n \]

where

(a) \( a_0(0) = a_1(0) = 1 \)
(b) \( a_n(0) = 0 \) for \( n \geq 2 \)
(c) \( a_n(y) = O(y^{\nu_n}) \) with \( \lim_{n \to \infty} \nu_n = \infty \)

and coefficients lie in a commutative ring-with-identity-element \( R \).

• There exists a unique formal power series \( x_0(y) \in R[[y]] \) satisfying \( f(x_0(y), y) = 0 \). We call \( x_0(y) \) the leading root of \( f \).

• Since \( x_0(y) \) has constant term \(-1\), we write \( x_0(y) = -\xi_0(y) \) where \( \xi_0(y) = 1 + O(y) \).

• We saw in Lecture #2 that \( \xi_0(y) \) can be computed by
  – An elementary method.
  – A method based on the explicit implicit function formula.
  – A method based on the exponential formula.
Method based on the explicit implicit function formula

• In Lecture #2 we derived the formula

\[
\frac{\xi_0(y)^\beta - 1}{\beta} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n_1, \ldots, n_m \geq 0} \left( \beta - 1 + \sum n_i \right) \prod_{i=1}^{m} (-1)^{n_i} \hat{a}_{n_i}(y)
\]

where

\[
\hat{a}_n(y) = \begin{cases} 
  a_n(y) - 1 & \text{for } n = 0, 1 \\
  a_n(y) & \text{for } n \geq 2
\end{cases}
\]

• Can this formula be used for proofs of nonnegativity???

• Recall the definition: \( \xi_0(y) \in S_\beta \) in case \( \frac{\xi_0(y)^\beta - 1}{\beta} \succeq 0 \) (coefficientwise nonnegativity)

• Empirically I know that \( \xi_0(y) \in S_\beta \) when \( a_n(y) = \alpha_n y^{n(n-1)/2} \) and

(a) \( \beta \geq -2 \) with \( \alpha_n = 1 \) (partial theta function)
(b) \( \beta \geq -1 \) with \( \alpha_n = 1/n! \) (deformed exponential function)
(c) \( \beta \geq -1 \) with \( \alpha_n = (1 - q)^n/(q; q)_n \) and \( q > -1 \)

• How can we see these facts from this formula???
  [open combinatorial problem]

• All these examples have \( \hat{a}_n(y) \succeq 0 \). The factors \( (-1)^{n_i} \) then seem to cause trouble.
A very simple case: Alternating signs

**Proposition.** Suppose that

\[ (-1)^n \hat{a}_n(y) \geq 0 \quad \text{for all } n \geq 0 \]

where

\[
\hat{a}_n(y) = \begin{cases} 
  a_n(y) - 1 & \text{for } n = 0, 1 \\
  a_n(y) & \text{for } n \geq 2 
\end{cases}
\]

Then \( \xi_0(y) \in S_\beta \) and in fact

\[
\frac{\xi_0(y)^\beta - 1}{\beta} \geq \sum_{n=0}^\infty (-1)^n \hat{a}_n(y)
\]

in the following cases:

(a) \( \beta = 1 \)
(b) \( \beta = -1 \) whenever \( a_0(y) = 1 \)
(c) \( \beta = -3 \) whenever \( a_0(y) = a_1(y) = 1 \)
(d) \( \beta = -(2k - 1) \) whenever \( a_0(y) = a_1(y) = 1 \) and \( a_2(y) = \ldots = a_{k-1}(y) = 0 \)

**Proof.** Follows almost immediately from

\[
\frac{\xi_0(y)^\beta - 1}{\beta} = \sum_{m=1}^\infty \frac{1}{m} \prod_{n_1, \ldots, n_m \geq 0} \left( \beta - 1 + \sum_{n_i(m-1)} \prod_{i=1}^m (-1)^{n_i} \hat{a}_{n_i}(y) \right)
\]

(a) Set \( \beta = 1 \). Then the RHS of the Proposition comes from the term \( m = 1 \). All the other terms are \( \geq 0 \) since \( (-1)^n \hat{a}_n(y) \geq 0 \) and \( \left( \beta - 1 + \sum_{n_i} \right) \geq 0 \).
(b) Set $\beta = -1$ and observe that the sum can be restricted to $n_1, \ldots, n_m \geq 1$. If $m = 1$ we have $\left(\beta - 1 + \sum n_i\right) = 1$ and we get the RHS of the Proposition. If $m \geq 2$ we have $\sum n_i \geq 2$, so that $\beta - 1 + \sum n_i$ is a nonnegative integer and hence $\left(\beta - 1 + \sum n_i\right) \geq 0$.

(c) is analogous to (b), but using $\beta = -3$ and observing that the sum can be restricted to $n_1, \ldots, n_m \geq 2$, so that $m \geq 2$ implies $\sum n_i \geq 4$.

(d) is analogous to (b), but using $\beta = -(2k - 1)$ and observing that the sum can be restricted to $n_1, \ldots, n_m \geq k$, so that $m \geq 2$ implies $\sum n_i \geq 2k$. 

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A slight strengthening (by rescaling of \( f \))

**Corollary.**

(a) If \((-1)^n \frac{a_n(y)}{a_1(y)} \geq 0\) for all \(n \neq 1\), then \(\xi_0(y) \in S_1\) and satisfies

\[
\xi_0(y) \geq \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)}.
\]

(b) If \(1 - \frac{a_1(y)}{a_0(y)} \geq 0\) and \((-1)^n \frac{a_n(y)}{a_0(y)} \geq 0\) for all \(n \geq 2\), then \(\xi_0(y) \in S_{-1}\) and satisfies

\[
\xi_0(y)^{-1} \leq \frac{a_1(y)}{a_0(y)} - \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_0(y)}.
\]

**Proof.**

(a) Apply part (a) of the Proposition to \(f(x, y)/a_1(y)\).

(b) Apply part (b) of the Proposition to \(f(x, y)/a_0(y)\).
Alternative (elementary) proof of the Corollary

• No need to use explicit implicit function formula. Just bare hands!

• **Proof of part (a):** Start from the equation $f(-\xi_0(y), y) = 0$, divide by $a_1(y)$, and add $\xi_0(y)$ to both sides:

$$\xi_0(y) = \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)} \xi_0(y)^n$$

• The unique solution to this equation can be found iteratively as follows: Define a map $F: \mathbb{R}[[y]] \rightarrow \mathbb{R}[[y]]$ by

$$(F\xi)(y) = \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)} \xi(y)^n$$

and define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{R}[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = F\xi_0^{(k)}$. I then claim that

$$\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \ldots \preceq \xi_0$$

and that

$$\xi_0^{(k)}(y) = \xi_0(y) + O(y^{k+1})$$

**Proof of claim:**

− If $f(y)$ and $g(y)$ are formal power series satisfying $0 \preceq f \preceq g$, then the hypotheses of the Corollary [part (a)] guarantee that $0 \preceq Ff \preceq Fg$.

− Applying this repeatedly to the obvious inequality $0 \preceq \xi_0^{(0)} \preceq \xi_0^{(1)}$, we obtain $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \ldots$. 


Likewise, if \( f(y) \) and \( g(y) \) are formal power series satisfying
\[ f(y) - g(y) = O(y^\ell) \]
for some \( \ell \geq 0 \), then it is easy to see that
\[ (\mathcal{F} f)(y) - (\mathcal{F} g)(y) = O(y^{\ell+1}) \]
[since \( a_n(y)/a_1(y) = O(y) \) for all \( n \geq 2 \)].

Applying this repeatedly to the obvious fact \( \xi^{(1)}_0(y) - \xi^{(0)}_0(y) = O(y) \), we obtain
\[ \xi^{(k+1)}_0(y) - \xi^{(k)}_0(y) = O(y^{k+1}). \]

It follows that \( \xi^{(k)}_0(y) \) converges as \( k \to \infty \) (in the topology of formal power series) to a limiting series \( \xi^{(\infty)}_0(y) \), and that this limiting series satisfies \( \mathcal{F} \xi^{(\infty)}_0 = \xi^{(\infty)}_0 \). But this means that \( \xi^{(\infty)}_0(y) = \xi_0(y) \). It also follows that \( \xi^{(k)}_0(y) = \xi_0(y) + O(y^{k+1}) \). The inequality of the Corollary is precisely the statement \( \xi_0 \geq \xi^{(1)}_0 \).

- The proof of part (b) is similar.
- Can parts (c) and (d) of the Proposition be given a similarly elementary proof?
- Can results analogous to the Proposition be proven for the spaces \( \mathcal{S}_\beta \) with \( \beta \neq 1, -1, -3, -5, \ldots \)?

But isn’t the case of alternating signs too trivial?

- After all, the most interesting examples have constant signs.
- Then the irritating factors \( (-1)^n \) cannot be avoided.
The partial theta function \( \Theta_0(x, y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2} \)
(which has \textit{constant signs}!)

It seems that \( \xi_0(y) \in S_1: \)
\[
\xi_0(y) = 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 + 948y^9 + 2610y^{10} + \ldots + \text{terms through order } y^{6999}
\]

and indeed that \( \xi_0(y) \in S_{-1}: \)
\[
\xi_0(y)^{-1} = 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 - 178y^9 - 490y^{10} - \ldots - \text{terms through order } y^{6999}
\]

and indeed that \( \xi_0(y) \in S_{-2}: \)
\[
\xi_0(y)^{-2} = 1 - 2y - y^2 - y^3 - 2y^4 - 7y^5 - 18y^6 - 50y^7 - 138y^8 - 386y^9 - \ldots - \text{terms through order } y^{6999}
\]

Can we prove any of this???

\textbf{Yes!!!} (for the first two properties; the third is still open)
Proof for the partial theta function

• Use standard notation for $q$-shifted factorials:

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) \quad \text{for } |q| < 1$$

• A pair of identities for the partial theta function:

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (y; y)_\infty (-x; y) \sum_{n=0}^{\infty} \frac{y^n}{(y; y)_n (-x; y)_n}$$

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (-x; y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n y^{n^2}}{(y; y)_n (-x; y)_n}$$

as formal power series and as analytic functions on $(x, y) \in \mathbb{C} \times \mathbb{D}$

• Rewrite these as

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (y; y)_\infty (-xy; y) \left[ 1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y; y)_n (-xy; y)_{n-1}} \right]$$

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (-xy; y)_\infty \left[ 1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n y^{n^2}}{(y; y)_n (-xy; y)_{n-1}} \right]$$

• Brackets on the RHS (minus the initial $1+x$) have alternating signs in $x$ (i.e. have nonnegative coefficients as a series in $-x$ and $y$)

• So we have reduced to the easy case of alternating signs!

• The second identity has $a_0(y) = 1$, so we prove also $\xi_0(y) \in \mathcal{S}_{-1}$. 

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The preceding proof, written more explicitly

• Let’s say we use the first identity:

\[
\Theta_0(x, y) = (y; y)_\infty (-xy; y)_\infty \left[ 1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y; y)_n (-xy; y)_{n-1}} \right]
\]

• So \( \Theta_0(x, y) = 0 \) is equivalent to “brackets = 0”.

• Insert \( x = -\xi_0(y) \) and bring \( \xi_0(y) \) to the LHS:

\[
\xi_0(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^{n} (1 - y^j) \prod_{j=1}^{n-1} [1 - y^j \xi_0(y)]}
\]

• This formula can be used iteratively to determine \( \xi_0(y) \), and in particular to prove the strict positivity of its coefficients:

• Define the map \( F: \mathbb{Z}[[y]] \to \mathbb{Z}[[y]] \) by

\[
(F\xi)(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^{n} (1 - y^j) \prod_{j=1}^{n-1} [1 - y^j \xi(y)]}
\]

• Define a sequence \( \xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[[y]] \) by \( \xi_0^{(0)} = 1 \) and \( \xi_0^{(k+1)} = F\xi_0^{(k)} \).

• Then \( \xi_0^{(0)} \leq \xi_0^{(1)} \leq \ldots \leq \xi_0 \) and \( \xi_0^{(k)}(y) = \xi_0(y) + O(y^{3k+1}) \).

• In particular, \( \lim_{k \to \infty} \xi_0^{(k)}(y) = \xi_0(y) \), and \( \xi_0(y) \) has strictly positive coefficients.

• Thomas Prellberg has a combinatorial interpretation of \( \xi_0(y) \) and \( \xi_0^{(k)}(y) \).
Elementary proof of the first identity

- Proof uses nothing more than Euler’s first and second identities

\[
\frac{1}{(t; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n}
\]

\[
(t; q)_{\infty} = \sum_{n=0}^{\infty} \frac{(-t)^n q^{n(n-1)/2}}{(q; q)_n}
\]

valid for \((t, q) \in \mathbb{D} \times \mathbb{D}\) and \((t, q) \in \mathbb{C} \times \mathbb{D}\), respectively.

- Write

\[
\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2} \frac{(y; y)_{\infty}}{(y; y)_n (y^{n+1}; y)_{\infty}}
\]

- Insert Euler’s first identity for \(1/(y^{n+1}; y)_{\infty}\):

\[
\Theta_0(x, y) = (y; y)_{\infty} \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(y; y)_n} \sum_{k=0}^{\infty} \frac{y^{(n+1)k}}{(y; y)_k}
\]

\[
= (y; y)_{\infty} \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k} \sum_{n=0}^{\infty} \frac{(xy^k)^n y^{n(n-1)/2}}{(y; y)_n}
\]

\[
= (y; y)_{\infty} \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k} (-x y^k; y)_{\infty} \quad \text{by Euler’s second identity}
\]

\[
= (y; y)_{\infty} (-x; y)_{\infty} \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k (-x; y)_k}
\]

- This identity goes back to Heine (1847), but does not seem to be very well known.

- It can be found in Fine (1988) and Andrews and Warnaar (2007).

- Did anyone know it between 1847 and 1988???
Proof of the first and second identities

- A simple limiting case of Heine’s first and second transformations

\[
2\phi_1(a, b; c; q, z) = \frac{(b; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty} 2\phi_1(c/b, z; az; q, b)
\]

\[
2\phi_1(a, b; c; q, z) = \frac{(c/a; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty} 2\phi_1(abz/c, a; az; q, c/a)
\]

for the basic hypergeometric function

\[
2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n
\]

- Just set \(b = q\) and \(z = -x/a\), then take \(a \to \infty\) and \(c \to 0\).
- This is how Heine (1847) proved the first identity.
- Heine didn’t know his second transformation, which is apparently due to Rogers (1893).

**Who first wrote the second identity for the partial theta function??**

- Surely it must have been known before Andrews and Warnaar (2007)!??
Can any of this be generalized?

- Recall our friend
  \[ \tilde{R}(x, y, q) = \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(1 + q) \cdots (1 + q + \ldots + q^{n-1})} \]

- Can this proof be extended to cases \( q \neq 0 \)?

- Here is a general identity:
  \[ \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} = \frac{1}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q; q)_\ell} \Theta_0(xq^\ell, y) \]

- Can deduce generalizations of the first and second identities for the partial theta function:
  \[ \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} = \frac{(y; y)_{\infty}}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q; q)_\ell} (-xq^\ell; y)_{\infty} \sum_{n=0}^{\infty} \frac{y^n}{(y; y)_n (-xq^\ell; y)_n} \]
  \[ \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} = \frac{1}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q; q)_\ell} (-xq^\ell; y)_{\infty} \sum_{n=0}^{\infty} \frac{(-xq^\ell)^n y^{n^2}}{(y; y)_n (-xq^\ell; y)_n} \]

- But I don’t know what to do with these formulae, because of the factors \((-1)^\ell\).