Block intersection polynomials
(and their applications to graphs and block designs)

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Block intersection polynomials (invented by Peter J. Cameron and LHS) give useful information on the feasible solutions to integer programming problems of a certain type which arise in the study of graphs and block designs having certain regularity properties.

I shall define block intersection polynomials, and give some examples of the theory of these polynomials and their applications to the studies of edge-regular graphs, amply regular graphs, and $t$-designs.

All graphs in this talk are finite and undirected, with no loops and no multiple edges.
Some definitions

• A graph $\Gamma$ is edge-regular with parameters $(v, k, \lambda)$ if $\Gamma$ has exactly $v$ vertices, is regular of degree $k$, and every pair of adjacent vertices have exactly $\lambda$ common neighbours.

• A graph is amply regular with parameters $(v, k, \lambda, \mu)$ if it is edge-regular with parameters $(v, k, \lambda)$ and every pair of vertices at distance 2 have exactly $\mu$ common neighbours.

• A graph is strongly regular with parameters $(v, k, \lambda, \mu)$ if it is edge-regular with parameters $(v, k, \lambda)$ and every pair of distinct nonadjacent vertices have exactly $\mu$ common neighbours (so in particular, every strongly regular graph is amply regular).
• A clique in a graph is a set of pairwise adjacent vertices.

• A block design is an ordered pair \((V, \mathcal{B})\), such that \(V\) is a finite non-empty set, whose elements are called points, and \(\mathcal{B}\) is a finite non-empty multiset of subsets of \(V\) called blocks.

• For \(t\) a non-negative integer and \(v, k, \lambda\) positive integers with \(t \leq k \leq v\), a \(t-(v, k, \lambda)\) design (or simply a \(t\)-design) is a block design with exactly \(v\) points, such that each block has size \(k\) and each \(t\)-subset of the point-set is contained in exactly \(\lambda\) blocks.

• The incidence graph of a block design \(D\) is the graph whose vertices are the points and blocks of \(D\) (including repeated blocks), with \(\{\alpha, \beta\}\) an edge precisely when one of \(\alpha\) and \(\beta\) is a point and the other is a block containing that point.
For example, the block design

\[ Z := (V, B) \]

with point set

\[ V := \{1, \ldots, 8\}, \]

and block multiset \( B := \)

\[ [1234, 1238, 1256, 1357, 1458, 1467, 1678, 2367, 2457, 2468, 2578, 3456, 3478, 3568] \]

is a 2-(8, 4, 3) design.
Now, let $\Gamma$ be a graph, and let $S$ and $Q$ be given vertex-subsets of $\Gamma$, with $s := |S|$.

We are interested in using regularity properties of $\Gamma$ and information on the subgraph induced on $S$ to obtain information about the number $n_i$ of vertices in $Q$ adjacent to exactly $i$ vertices in $S$ ($i = 0, \ldots, s$), sometimes with the aim of obtaining a contradiction to show that no triple $(\Gamma, S, Q)$ can exist with the given properties.
For $T \subseteq S$, define $\lambda_T$ to be the number of vertices in $Q$ adjacent to every vertex in $T$, and for $0 \leq j \leq s$, define

$$
\lambda_j := 1/\binom{s}{j} \sum_{T \subseteq S, |T| = j} \lambda_T.
$$

For example, if $\Gamma$ is an edge-regular graph with parameters $(v, k, \lambda)$, $S$ an $s$-clique of $\Gamma$ with $s \geq 2$, and $Q := V(\Gamma) \setminus S$, then

$$
\lambda_0 = v - s, \quad \lambda_1 = k - s + 1, \quad \lambda_2 = \lambda - s + 2.
$$

For another example, if $\Gamma$ is the incidence graph of a $t$-$(v, k, \lambda)$ design $D$, $S$ the set of vertices of $\Gamma$ consisting of the points on some block $B$ of $D$, and $Q$ the set of vertices of $\Gamma$ corresponding to the blocks of $D$, then $n_i$ is the number of blocks of $D$ meeting $B$ in exactly $i$ points, and for $j = 0, \ldots, t$,

$$
\lambda_j = \lambda_j(D) = \lambda \binom{v-j}{t-j} / \binom{k-j}{t-j},
$$

the (constant) number of blocks of $D$ containing a $j$-subset of the point-set.
For each known $\lambda_j$, we have the equation:

$$\sum_{i=0}^{s} \binom{i}{j} n_i = \binom{s}{j} \lambda_j. \quad (1)$$

**Theorem (with PJC)** For $k$ a non-negative integer, define the polynomial

$$P(x, k) := x(x-1)\cdots(x-k+1),$$

let $s$ and $t$ be integers, with $s \geq t \geq 0$, let $n_0, \ldots, n_s$, $m_0, \ldots, m_s$, and $\lambda_0, \ldots, \lambda_t$ be real numbers, and suppose that for $j = 0, \ldots, t$, equation (1) holds. Then

$$\sum_{i=0}^{s} P(i-x,t)(n_i - m_i) =$$

$$\sum_{j=0}^{t} \binom{t}{j} P(-x, t-j)[P(s,j)\lambda_j - \sum_{i=j}^{s} P(i,j)m_i]. \quad (2)$$

We call (2) the *block intersection polynomial* for the sequences $[m_0, \ldots, m_s]$, $[\lambda_0, \ldots, \lambda_t]$, and denote this polynomial by

$$B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t]).$$
The preceding theorem can be applied to prove:

**Theorem** Let $\Gamma$ be a graph, let $S$ and $Q$ be vertex-subsets of $\Gamma$, with $s := |S|$, and let $m_0, \ldots, m_s$ be non-negative integers with either $m_i \leq n_i$ for all $i$ or $m_i \geq n_i$ for all $i$, where $n_i$ is the number of vertices in $Q$ adjacent to exactly $i$ vertices in $S$.

Let $t$ be an **even** integer with $0 \leq t \leq s$, and for $j = 0 \ldots, t$, let $\lambda_j := 1/\binom{s}{j} \sum_{T \subseteq S, |T| = j} \lambda_T$, where $\lambda_T$ is the number of vertices in $Q$ adjacent to every vertex in $T$.

Now, let $B(x) := B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t])$. Then:
• \( B(x) \equiv 0 \) if and only if \( m_i = n_i \) for all \( i \); otherwise, \( B(x) \) is a degree \( t \) polynomial with integer coefficients.

• \( B(m) \geq 0 \) for every integer \( m \) if \( m_i \leq n_i \) for all \( i \), and \( B(m) \leq 0 \) for every integer \( m \) if \( m_i \geq n_i \) for all \( i \).

• \( B(m) = 0 \) for some integer \( m \) if and only if \( m_i = n_i \) for all \( i \notin \{m, m+1, \ldots, m+t-1\} \), in which case \([n_0, \ldots, n_s]\) is uniquely determined by \([m_0, \ldots, m_s]\) and \([\lambda_0, \ldots, \lambda_t]\).
Example of bounding clique-size in an edge-regular graph

The strongly regular graphs with parameters $(37, 18, 8, 9)$ include Paley(37), but not all strongly regular graphs with these parameters are known. The complement of such a graph (and such a graph) has least eigenvalue $\tau \approx -3.541$, and so the Hoffman bound gives an upper bound of $6 = \lfloor 37/(1 - 18/\tau) \rfloor$ on the size of a clique.

Now let $\Gamma$ be any edge-regular graph with parameters $(37, 18, 8)$, and suppose that $\Gamma$ contains a clique $S$ of size 6. We calculate $B(x) := B(x, [0^7], [31, 13, 4]) = 31x^2 - 125x + 120$, and find that $B(2) = -6$. Hence $\Gamma$ contains no clique of size 6.

I do not know whether there is some edge-regular graph with parameters $(37, 18, 8)$ and a clique of size 5. The size of a maximum clique in Paley(37) is 4.
Application to amply regular graphs

**Theorem** Let $\Gamma$ be an amply regular graph with parameters $(v, k, \lambda, \mu)$, and suppose $\Delta$ is an induced subgraph of $\Gamma$, where $\Delta$ has $s \geq 2$ vertices and vertex-degree sequence $[d_1, \ldots, d_s]$. Further suppose that $\Delta$ is connected with diameter at most 2 if $\Gamma$ is not strongly regular. Let $B(x) := x(x + 1)(v - s) - 2xsk + (2x + \lambda - \mu + 1) \sum_{i=1}^{s} d_i + s(s - 1)\mu - \sum_{i=1}^{s} d_i^2$.

Then $B(m) \geq 0$ for every integer $m$.

Moreover, $B(m) = 0$ for some integer $m$ if and only if each vertex not in $\Delta$ is adjacent to exactly $m$ or $m + 1$ vertices of $\Delta$, in which case exactly $B(m+1)/2$ vertices not in $\Delta$ are adjacent to just $m$ vertices of $\Delta$. 
Example

Let $\Gamma$ be a strongly regular graph with parameters $(76, 30, 8, 14)$. It is unknown whether such a graph exists, although these are “feasible” parameters for a strongly regular graph.

Now suppose $\Gamma$ contains an induced subgraph $\Delta$ isomorphic to (the 1-skeleton of) an octahedron, i.e. the strongly regular graph with parameters $(6, 4, 2, 4)$. Then $\Delta$ has $s = 6$ vertices and vertex-degree sequence $[4^6]$. We calculate $B(x)$ as in the Theorem above, and determine that

$$B(x) = 70(x - 2)(x - 51/35).$$

In particular, $B(2) = 0$. Hence, exactly $B(3)/2 = 54$ vertices not in $\Delta$ are adjacent to exactly 2 vertices of $\Delta$, and the remaining 16 vertices not in $\Delta$ are adjacent to exactly 3 vertices of $\Delta$. 
Example of bounding the multiplicity of a block in a \( t \)-design

Suppose \( D \) is a 4-(23, 8, 6) design (designs with these parameters exist). Further suppose that \( D \) has a block \( B \) of multiplicity 3 or more. Then there are at least 3 blocks meeting \( B \) in 8 points.

Now let

\[
\Lambda := [\lambda_0(D), \ldots, \lambda_4(D)] = [759, 264, 84, 24, 6],
\]

and calculate

\[
B(x) := B(x, [0^8, 3], \Lambda)
\]

\[
= 36(21x^4 - 106x^3 + 291x^2 - 366x + 140).
\]

Since \( B(1) = -720 \), we conclude it is impossible for a block of \( D \) to have multiplicity 3 or more, and so each block of a 4-(23, 8, 6) design can have multiplicity at most 2.

This also shows that each block of a 5-(24, 9, 6) design (such designs exist) can have multiplicity at most 2.
Example for a resolvable $t$-design

It is unknown whether there exists a 2-(55, 11, 5) design, but we can show that in such a design, each block has multiplicity at most 2.

Suppose now $D$ is a resolvable 2-(55, 11, 5) design. (A block design is resolvable if its blocks can be partitioned into parallel classes, a parallel class being a set of blocks partitioning the point set.) Further suppose that $D$ has a block $B$ of multiplicity 2 or more. Then there are at least 2 blocks meeting $B$ in 11 points and at least 8 blocks meeting $B$ in no points.

Now let

$$\Lambda := [\lambda_0(D), \lambda_1(D), \lambda_2(D)] = [135, 27, 5],$$

and calculate

$$B(x) := B(x, [8, 0^{10}, 2], \Lambda) = 5(25x^2 - 85x + 66).$$

Since $B(2) = -20$, we conclude that no block of a resolvable 2-(55, 11, 5) design has multiplicity 2 or more. In other words, each resolvable 2-(55, 11, 5) design is simple.
Finally, here is a new theoretical application of block intersection polynomials to the study of $t$-designs.

**Theorem** Let $t$ be an even positive integer, let $D$ be a $t-(v, k, \lambda)$ design, and for $B$ a block of $D$, define $I(D, B)$ to be the set of all $i$ for which some block of $D$, other than $B$, meets $B$ in exactly $i$ points. Now suppose that for some block $B$ of $D$, $I(D, B)$ is contained in a set of $t$ consecutive integers.

Then for every $t-(v, k, \lambda)$ design $E$, every block $C$ of $E$, and every $i = 0, \ldots, k$, the number of blocks of $E$ meeting $C$ in exactly $i$ points is the same as the number of blocks of $D$ meeting $B$ in exactly $i$ points.
In some sense, this result is best possible, for consider the $2-(8, 4, 3)$ design $Z$ given at the beginning of this talk.

The sizes of the intersections of the block 1234 with the other blocks of $Z$ are the three consecutive integers 1, 2, 3, and the sizes of the intersections of the block 1357 with the other blocks of $Z$ are the two nonconsecutive integers 0, 2.
For details, generalizations, proofs, and computer implementations, see:


L.H. Soicher, More on block intersection polynomials and new applications to graphs and block designs, available from

http://designtheory.org/library/preprints/

L.H. Soicher, The DESIGN package for GAP, Version 1.3, 2006,

http://designtheory.org/software/gap_design/