Matroid Representation Over Infinite Fields

General Theme

There exist strong theorems for matroids representable over finite fields, but either

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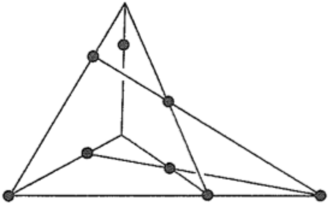
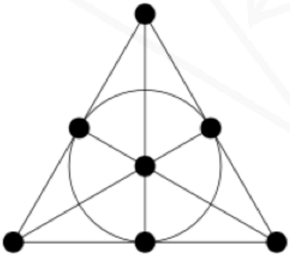
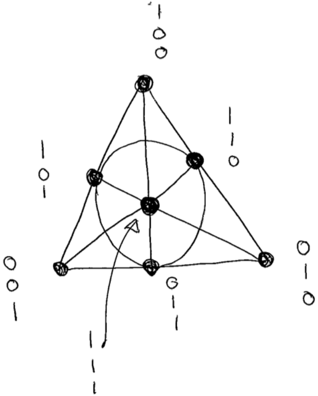
- In this talk "the reals" will frequently be code for any infinite field.

A Primer on Matroids

- Matroids capture the *combinatorial* properties of a finite collection of points in “space”.
- A natural way to describe a matroid is as a finite set of vectors in a vector space.
- We can choose any field we like.
- A matroids is *representable* over a field if it can be described via a set of vectors over that field.



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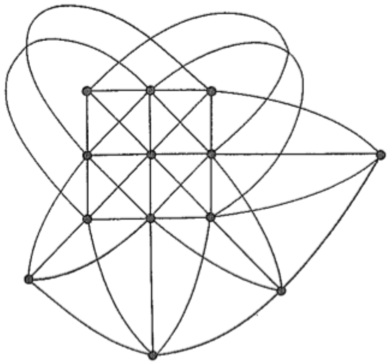


Matroid Axioms

A matroid M is a set S together with a collection $\mathcal{I}$ of subsets of S such that

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
- If $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists $z \in J - I$ such that $I \cup \{z\} \in \mathcal{I}$.

The *rank* of a set A is the size of a maximal independent set in A.

Substructure

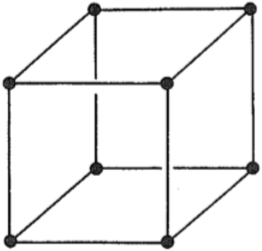
Can remove elements from a matroid by

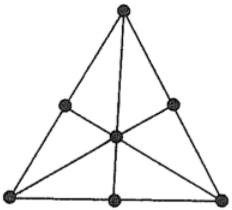
- **Deletion** Simply remove them from consideration.
- **Contraction** Corresponds to projecting from the element.

Minor

A minor is obtained by a sequence of deletions and contractions.

Minors are the fundamental notion of substructure in matroids.





Minor-closed Classes

A class $\mathcal{M}$ of matroids is minor closed if whenever $M \in \mathcal{M}$, so too is any minor of M.

Excluded Minors

An excluded minor for $\mathcal{M}$ is a matroid not in $\mathcal{M}$ all of whose proper minors are in $\mathcal{M}$.

The 4-point line is an excluded minor for $GF(2)$.

Theorem (Tutte 1956)

A matroid is binary if and only if it has no 4-point line minor.

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Rota's Conjecture

For any fixed finite field $\mathbb{F}$ there are a finite number of forbidden minors for $\mathbb{F}$ representability.

Easy Exercise

There are infinitely many excluded minors for matroids representable over any fixed infinite field.

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Theorem (Mayhew, Newman, W)

For any real-representable matroid M, there is an excluded minor for real representability that contains M as a minor.

What is a well-quasi-order?

- A quasi-order $\preceq$ on a set X is a reflexive, transitive relation on X.
- Quasi orders are essentially partial orders.
- An antichain in a quasi-order is a set of pairwise incomparable elements.

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- A quasi-order $\preceq$ on a set X is a reflexive, transitive relation on X.
- Quasi orders are essentially partial orders.
- An antichain in a quasi-order is a set of pairwise incomparable elements.
- A well-quasi-order has no infinite antichains.

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- $12, 16, 100$ is an antichain.
- Do we have a well-quasi-order?
- No. There are infinitely many primes.

Well-quasi-ordering Conjecture

Matroids over a finite field are well-quasi-ordered.

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- Matroids over an infinite field are not.

Minor-testing Conjecture

Can recognise any minor-closed property in polynomial time for matroids representable over a finite field.

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- Cannot recognise uniform matroids over the reals.

Deciding Representability

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- This extends easily to any other field, finite or infinite.

Certifying non-representability

Theorem (Geelen, W.)

It requires only a polynomial number of calls to a rank oracle to prove that a matroid is not representable over a prime field.

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- (ben David and Geelen) It requires exponentially many calls to prove that M is not representable over the reals.

Branch width and tree width

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"The fundamental problems of deciding which matroids are matrix is left unsolved."

- Whitney almost certainly had real representable matroids in mind.
- Search for the missing axiom of matroid theory!

The Rank Axioms

E a finite subset of $\mathbb{R}^n$. For $A \subseteq E$, the *rank* of A, denoted $r(A)$, is the size of a max independent subset of A. We have:

- R1 $r(\emptyset) = 0$.
- R2 If $e \in E$, then $0 \leq r(\{e\}) \leq 1$.
- R3 If $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.
- R4 If $A, B \subseteq E$, then $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$.

A *matroid* is a finite set E together with a function $r : 2^E \to \mathbb{Z}$ satisfying R1, R2, R3 and R4.

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No $U_{2,4}$ minor is equivalent to

R5 For all $X \subseteq E$, it is not that case that there exists $Y \subseteq E - X$ with $|Y| = 4$ such that for all $Z \subseteq Y$, $r(X \cup Z) = |X| + |Z|$ if $|Z| \leq 2$, and otherwise $r(X \cup Z) = r(X) + 2$.

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We've found the missing axiom of binary matroids!

Theorem

A matroid is binary if and only if it satisfies R1, R2, R3, R4 and R5.

Vamos 1978 paper. "The missing axiom of matroid theory is lost forever."

Theorem (Vamos)

It is not possible to add a finite number of axioms expressed in first order logic *to the matroid axioms to characterise real representability.*

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- But the proof only needs the fact that these are forbidden submatroids.
- Binary matroids have an infinite number of forbidden submatroids, ie $U_{n,n+2}$ for all $n \geq 2$.
- Therefore Vamos' proof works for binary matroids!

My Favourite Conjecture

Let $\mathcal{R}$ be the set of real representable matroids and $\mathcal{R}^+$ be the set of real representable matroids together with the set of excluded minors for real representability.

Conjecture (Mayhew, Newman, W.)

For all $\epsilon > 0$, there is an N such that if $n > N$, then the proportion of n-element members of $\mathcal{R}^+$ that are in $\mathcal{R}$ is less than ϵ.

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This leads to the intriguing subject of Fractal Classes of Matroids.

Typical Properties

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We know essentially nothing about typical properties of matroids representable over infinite fields.

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Dillon Mayhew, Mike Newman, Dominic Welsh and Geoff Whittle, On the asymptotic proportion of connected matroids, European Journal of Combinatorics 32 (2011) 882890

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- We come to the world preprogrammed to see it in a certain way.
- Euclidean Geometry is synthetic a priori.
- Real representable matroids are synthetic a priori.

Conclusion

Real-representable matroids are the most natural matroids from the perspective of human intuition. Yet, when viewed through a matroid-theorist's lens they become the most fundamentally mysterious of all classes.