

# Partition backtrack methods for more complicated group actions

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## 1 Outline

- **Combinatorial isomorphism** problems: action of group  $G$  on set  $\Omega$ .
- **Current implementation** requires  $G = S_n$ .
- Not even possible to use an **arbitrary permutation group**.
- **Two goals**
  - **Augment** the framework for added flexibility.
  - **Generalise** the framework to work with general actions.
- **add bells and whistles** vs. **generalise** existing bells and whistles.
- **Progress** has been made on both.
- This talk: **generalisation**.

## 2 Generalisation: what, why, and how

### 2.1 What?

- **Two clear ways:**
  - Generalise the types of objects used to define permutations.
  - Generalise the type of group that acts on the objects.

- **Progress** has been made on the **latter**.
- Work is **very new** and certainly incomplete.
- When finished should provide a “**shopping list**” of functions.

## 2.2 Why?

- **Why generalise** to groups other than  $S_n$ ?
  - **At least** deal with different permutation groups.
  - **Ideally** deal with abstract groups.
- **E.g.:** Different actions on the set  $\mathbb{F}_q^{[n]}$  of length  $n$  codewords.

1. Act on  $\mathbb{F}_q^{[n]}$  with  $S_n$ :

$$\begin{aligned} S_n \times \mathbb{F}_q^{[n]} &\rightarrow \mathbb{F}_q^{[n]} \\ (x, \omega) &\mapsto \omega \circ x^{-1} \end{aligned}$$

2. Act on  $\mathbb{F}_q^{[n]}$  with  $\mathbb{F}_q^\times$  wr  $S_n$ :

$$\begin{aligned} \left( (\mathbb{F}_q^\times)^{[n]} \times S_n \right) \times \mathbb{F}_q^{[n]} &\rightarrow \mathbb{F}_q^{[n]} \\ ((u, x), \omega) &\mapsto u \cdot (\omega \circ x^{-1}). \end{aligned}$$

3. Act on  $\mathbb{F}_q^{[n]}$  with  $\text{Aut}(\mathbb{F}_q) \times (\mathbb{F}_q^\times \text{ wr } S_n)$ :

$$\begin{aligned} \left( \text{Aut}(\mathbb{F}_q) \times \left( (\mathbb{F}_q^\times)^{[n]} \times S_n \right) \right) \times \mathbb{F}_q^{[n]} &\rightarrow \mathbb{F}_q^{[n]} \\ ((\lambda, (u, x)), \omega) &\mapsto \lambda(u \cdot (\omega \circ x^{-1})) \end{aligned}$$

- The **second and third** actions on  $\mathbb{F}_q^{[n]}$  certainly don't seem isomorphic to symmetric groups.
- $\therefore$  current algorithm **falls short** of **naturally** working with sophisticated group actions.

## 2.3 How?

- Many mechanisms **dependent** on ordered partitions.
- $\therefore$  Want to keep using them.
- **But:** To move past using  $S_n$  we need to move past  $[n]$ .
- Only concept dependent on  $[n]$ : the **group action**.
- All else depends on concepts that **generalise readily**.

## 3 Catalogue what we have

### 3.1 Objects

- Set of objects  $\Omega$ .
- $S_n$  acts on left of  $\Omega$ .

### 3.2 Points

- Set of points  $[n]$ .
- $S_n$  acts **faithfully** on left of  $[n]$ .

### 3.3 Partitions on points

- Set  $\Pi_n$  of **ordered partitions** over  $[n]$ .
- $S_n$  acts **pointwise** on the left of  $\Pi_n$ .
- **Two critical aspects**.
- **Firstly:** each  $\pi$  in  $\Pi_n$  **defines subset** of  $S_n$ .
  - All perms that send  $\pi$  to a **canonical representative**  $h(\pi)$ .
  - Obtained with function

$$\begin{aligned} \mathcal{B} : \Pi_n &\rightarrow \Pi_n \\ \pi &\mapsto \{x \in S_n \mid x\pi = h(\pi)\} \end{aligned}$$

- **Harmonious partition** is canonical representative.
- **Secondly:** we construct a **refinement process**.
  - Gives a set of fine partitions for each  $(\alpha, \pi) \in \Omega \times \Pi_n$ .
  - Each **fine partition** gives **exactly one permutation**.
  - $\therefore$  a set of permutations is found.
- Properties of  $\mathcal{B}$  and refinement  $\Rightarrow$  **all automorphisms** and **canonical representative** obtained.
- What are these properties?

## 4 Distill the essentials

- **Two fundamental actions** of  $S_n$ : one on  $\Omega$  and one on  $[n]$ .
- All other  $G$  actions are **built up**.
- **Roughly:** Studying the (**probably complicated**) action of  $S_n$  on  $\Omega$  by working with the (**hopefully simpler**) action of  $S_n$  over  $[n]$ .
- **Two stages** (concurrent in practice).
- **Firstly:** Generate (refine) ordered partitions.
  - **Critical:** All **functions** involved in **refinement** are  $S_n$ -morphisms.
  - Split:
 
$$\mathcal{S} : \Pi_n \times [n] \rightarrow \Pi_n.$$
  - Choose:
 
$$\mathcal{C} : \Omega \times \Pi_n \setminus \Phi_n \rightarrow \mathcal{P}([n]).$$
  - Refine:
 
$$\mathcal{R} : \Omega \times \Pi_n \rightarrow \Pi_n.$$
  - Tree:
 
$$\mathcal{T} : \Omega \times \Pi_n \rightarrow \mathcal{P}(\Omega \times \Pi_n).$$
- **Secondly:** Generate corresponding permutations.

- **Critical:** Function  $h$  is a **canonical map** over  $\Pi_n$  under  $S_n$ .
- **Practical need:** easily derive  $h(\pi)$  and  $\mathcal{B}(\pi)$ .
- **Result:** Leaf permutation function

$$\mathcal{L} : \mathcal{P}(\Omega \times \Pi_n) \rightarrow \mathcal{P}(S_n)$$

is an  $S_n$ -morphism.

## 5 Generalise

### 5.1 $G$ -spaces

- Act with an arbitrary group  $G$ .
- $\Omega$  is a  $G$ -space.
- Instead of  $[n]$  use **finite set**  $\Gamma$  forming a left  $G$ -space.
- Resulting set of ordered partitions:  $\Pi_\Gamma$ .
- Fine partitions:  $\Phi_\Gamma$ .
- $\Pi_\Gamma$  forms a left  $G$ -space under pointwise action.

### 5.2 Defining the subsets of $G$

- Analogous to  $S_n$  on  $\Pi_n$ .
- **Essential:** Canonical map

$$\mu : \Pi_\Gamma \rightarrow \Pi_\Gamma.$$

- (What used to be  $h$ )
- Recall:  $\forall \pi \in \Pi_\Gamma, \forall g \in G$

$$\mu(\pi) = \mu(g\pi)$$

and  $\exists g' \in G$  s.t.

$$\mu(\pi) = g'\pi.$$

- Want elements of  $G$  that send  $\pi$  to its **canonical representative**  $\mu(\pi)$ .

- **Defn:**

$$\begin{aligned}\mathcal{B} : \Pi_\Gamma &\rightarrow \mathcal{P}(G) \\ \pi &\mapsto \{g \in G \mid g\pi = \mu(\pi)\}\end{aligned}$$

- **Prop:**  $(\forall \pi \in \Pi_\Gamma) (\forall g \in \mathcal{B}(\pi))$

$$\mathcal{B}(\pi) = g\text{Aut}(\pi)$$

- **Corr:**  $\forall \pi \in \Pi_\Gamma, \forall g \in G$

$$\mathcal{B}(g\pi) = \mathcal{B}(\pi)g^{-1}$$

- So  $\mathcal{B}$  is a  $G$ -morphism.
- **Only one** group element from a **fine** partition?
- **Corr:** Let  $\pi$  be fine.  $|\mathcal{B}(\pi)| = 1$  if and only if  $G$  acts faithfully on  $\Gamma$ .
- **Proof:**

– **Definition**  $G$  is faithful on  $\Gamma \iff \bigcap_{\gamma \in \Gamma} \text{Aut}(\gamma) = \{1\}$ .

– **Since  $\pi$  is fine**  $\text{Aut}(\pi) = \bigcap_{\gamma \in \Gamma} \text{Aut}(\gamma)$ .

– **By Prop**  $|\mathcal{B}(\pi)| = 1 \iff \text{Aut}(\pi) = \{1\}$ .

- **Tree function** gives a subset of  $\Omega \times \Pi_\Gamma$ .
- Need to **derive** the **group elements** corresponding to fine partitions.
- **Defn:**

$$\begin{aligned}\mathcal{L} : \mathcal{P}(\Omega \times \Pi_\Gamma) &\rightarrow \mathcal{P}(G) \\ A &\mapsto \{g \in G \mid (\exists (\alpha, \pi) \in A) \pi \in \Phi_\Gamma \wedge g \in \mathcal{B}(\pi)\}\end{aligned}$$

- **Prop:**  $\mathcal{L}$  is a  $G$ -morphism; that is,  $(\forall A \in \mathcal{P}(\Omega \times \Pi_\Gamma)) (\forall g \in G)$

$$\mathcal{L}(gA) = \mathcal{L}(A)g^{-1}.$$

### 5.3 Refining partitions of $\Pi_\Gamma$

- **Refinement relation**  $\sqsubseteq$  can still be defined.
- **Same approach:** split, choose, and refine.

– Split:

$$\mathcal{S} : \Pi_\Gamma \times \Gamma \rightarrow \Pi_\Gamma.$$

– Choose:

$$\mathcal{C} : \Omega \times \Pi_\Gamma \setminus \Phi_\Gamma \rightarrow \mathcal{P}(\Gamma).$$

– Refine:

$$\mathcal{R} : \Omega \times \Pi_\Gamma \rightarrow \Pi_\Gamma.$$

- All **domains and codomains** are  $G$ -spaces
- We require all these functions to be  $G$ -**morphisms**.
- Definition of the **tree function** is identical to before.
- However,

$$\mathcal{T} : \Omega \times \Pi_\Gamma \rightarrow \mathcal{P}(\Omega \times \Pi_\Gamma)$$

- Induction is still useable; therefore,
  - $\mathcal{T}$  is a  $G$ -morphism
  - $\mathcal{T}$  always gives a set with at least one fine partition.

### 5.4 Combining the two

- $\mathcal{L} \circ \mathcal{T}$  will be a  $G$ -morphism.

- **Prop:**

$$(\forall (\alpha, \pi) \in \Omega \times \Pi_\Gamma) (\forall g \in G)$$

$$(\mathcal{L} \circ \mathcal{T}(g\alpha, g\pi))(g\alpha, g\pi) = (\mathcal{L} \circ \mathcal{T}(\alpha, \pi))(\alpha, \pi).$$

- **Prop:**

$$(\forall (\alpha, \pi) \in \Omega \times \Pi_\Gamma) (\forall g \in \mathcal{L} \circ \mathcal{T}(\alpha, \pi))$$

$$g\text{Aut}((\alpha, \pi)) \subseteq \mathcal{L} \circ \mathcal{T}(\alpha, \pi).$$

## 6 Outstanding tasks

- Still need to get at a **base and strong generating set**.
- **May impose** a few extra **limitations** on what function  $\mu$  can be.
- Nice interaction with the refinement relation may be one such need.
- **Examples** of functions that fit the shopping list are needed.