

# Graphs and finite transformation monoids

pjc, January 2009

This note describes a pair of mappings between graphs and transformation monoids on the set  $\{1, \dots, n\}$ , and some of their properties.

A *homomorphism* from a graph  $X$  to a graph  $Y$  is a map from the vertex set of  $X$  to that of  $Y$  which maps edges to edges; its behaviour on non-edges is unspecified. An *endomorphism* of  $X$  is a homomorphism from  $X$  to  $X$ . The set endomorphisms of a graph  $X$ , of course, is closed under composition and contains the identity; that is, it forms a monoid  $\text{End}(X)$ . An endomorphism is an *automorphism* if it is bijective. The automorphisms of  $X$  form a permutation group (a subgroup of the symmetric group  $S_n$ ), the *automorphism group* of  $X$ , denoted  $\text{Aut}(X)$ .

Two graphs  $X$  and  $Y$  are *hom-equivalent* if there are homomorphisms in both directions between them. A graph  $X$  is a *core* if there is no graph on fewer vertices which is hom-equivalent to  $X$ . Every graph  $X$  is hom-equivalent to a unique core, called the *core of  $X$* , and written  $\text{Core}(X)$ . The core of  $X$  has an embedding into  $X$ , and there is a retraction (a homomorphism which is the identity on its image) from  $X$  to the image of the embedding. We can recognise a core by the following property:

**Fact 1** *The graph  $X$  is a core if and only if  $\text{End}(X) = \text{Aut}(X)$ .*

The *clique number*  $\omega(X)$  of  $X$  is the size of the largest complete subgraph of  $X$ ; the *chromatic number*  $\chi(X)$  is the smallest number of colours required for a proper colouring of  $X$ . A class of graphs which will be very important in the sequel are those having the properties of the following result:

**Fact 2** *The graph  $X$  has the property that the core is complete if and only if  $\omega(X) = \chi(X)$ .*

**Proof**  $\omega(X) = m$  if and only if there is a homomorphism  $K_m \rightarrow X$ , while  $\chi(X) = m$  if and only if there is a homomorphism  $X \rightarrow K_m$ . So  $\omega(X) = \chi(X) = m$  if and only if  $X$  is homomorphically equivalent to  $K_m$  (which is then necessarily the core of  $X$ ).

In the other direction, let  $M$  be a transformation monoid on  $\{1, \dots, n\}$ , a submonoid of the full transformation monoid  $T_n$ . From  $M$ , we construct a graph as follows. Its vertex set is  $\{1, \dots, n\}$ ; two vertices  $i$  and  $j$  are joined by an edge if and only if there is no element  $f \in M$  for which  $if = jf$ . Denote this graph by  $\text{Gr}(M)$ .

A transformation monoid is *synchronizing* if it contains an element whose image has cardinality 1.

**Fact 3** (a)  $\text{Gr}(M)$  is complete if and only if  $M$  is a permutation group (that is, contained in the symmetric group).

(b)  $\text{Gr}(M)$  is null if and only if  $M$  is synchronizing.

**Proof** (a)  $\text{Gr}(M)$  is complete if and only if no element of  $M$  ever maps two points to the same place.

(b) Let  $f \in M$  be an element whose image is as small as possible. Then no two elements of the image of  $f$  can be mapped to the same place; so they are pairwise adjacent. So, if  $\text{Gr}(M)$  is null, then the image of  $f$  has cardinality 1. The converse is clear.

**Fact 4** For any transformation monoid  $M$ , the graph  $\text{Gr}(M)$  has core a complete graph.

**Proof** The argument in (b) above shows that the image of an element of  $M$  of minimal rank is a complete subgraph of  $\text{Gr}(M)$ . It is hom-equivalent to  $\text{Gr}(M)$  (the homomorphism in the other direction is just the embedding), and it is clearly a core.

**Fact 5** For any transformation monoid  $M$ ,

(a)  $M \leq \text{End}(\text{Gr}(M))$ ;

(b)  $\text{Gr}(\text{End}(\text{Gr}(M))) = \text{Gr}(M)$ .

**Proof** (a) Let  $f$  be an endomorphism of  $M$ , and let  $i$  and  $j$  be adjacent in  $\text{Gr}(M)$ . By definition,  $if \neq jf$ . Could  $if$  and  $jf$  be non-adjacent in  $\text{Gr}(M)$ ? if so, then there is an element  $h \in \text{End}(M)$  with  $(if)h = jf(h)$ . But this contradicts the adjacency of  $i$  and  $j$ , since  $fh \in M$  by closure.

(b) Suppose first that  $i$  and  $j$  are adjacent in  $\text{Gr}(M)$ . Then no endomorphism of  $\text{Gr}(M)$  can collapse them, so they are adjacent in  $\text{Gr}(\text{End}(\text{Gr}(M)))$ .

Conversely, suppose that  $i$  and  $j$  are not adjacent in  $\text{Gr}(M)$ . Then there is an element  $f \in M$  satisfying  $if = jf$ . By (a),  $f \in \text{End}(\text{Gr}(M))$ , and so  $i$  and  $j$  are non-adjacent in  $\text{Gr}(\text{End}(\text{Gr}(M)))$ .

It is not true that  $\text{End}(\text{Gr}(\text{End}(X))) = \text{End}(X)$  for all graphs  $X$ . For let  $X$  be the path of length 3, with just two automorphisms. It is easy to see that no endomorphism can identify the ends of the path, so that  $\text{Gr}(\text{End}(X))$  is the 4-cycle, with eight automorphisms.

**Fact 6** *The maps  $M \mapsto \text{End}(\text{Gr}(M))$  and  $X \mapsto \text{Gr}(\text{End}(X))$  are idempotent.*

**Proof** This follows immediately from part (b) of the preceding Fact.

Write  $\text{Cl}(M) = \text{End}(\text{Gr}(M))$ . Then  $M \leq \text{Cl}(M)$  and  $\text{Cl}(\text{Cl}(M)) = \text{Cl}(M)$ , so  $\text{Cl}$  is a closure operator on transformation monoids on  $\{1, \dots, n\}$ . I don't have a satisfactory description of the closed objects; but more on this below.

In the other direction, let  $\text{Hull}(X) = \text{Gr}(\text{End}(X))$ , so that  $\text{Hull}(\text{Hull}(X)) = \text{Hull}(X)$ . The hull of a graph has the following properties:

**Fact 7** (a)  *$X$  is a spanning subgraph of  $\text{Hull}(X)$  (that is, these graphs have the same vertex set, and every edge of  $X$  is an edge of  $\text{Hull}(X)$ ).*

(b)  $\text{End}(X) \leq \text{End}(\text{Hull}(X))$  and  $\text{Aut}(X) \leq \text{Aut}(\text{Hull}(X))$ .

(c)  $\text{Core}(\text{Hull}(X))$  is a complete graph on the vertex set of  $\text{Core}(X)$ .

**Proof** (a) If  $i$  and  $j$  are adjacent in  $X$ , then no endomorphism of  $X$  can collapse  $i$  and  $j$ , so they are adjacent in  $\text{Gr}(\text{End}(X))$ .

(b) Immediate from Fact 5(a).

(c) The vertex set of  $\text{Core}(X)$  cannot be collapsed by endomorphisms, so is a complete subgraph of  $\text{Gr}(\text{End}(X)) = \text{Hull}(X)$ .

By (c), if  $X$  is a hull, then  $\text{Core}(X)$  is complete; but the converse is false. If  $X$  is the path of length 3, then  $\text{Core}(X)$  is a complete graph on two vertices, but  $\text{Hull}(X)$  is the 4-cycle, by our previous argument.

**Fact 8** *A transformation monoid  $M$  is closed (that is, satisfies  $M = \text{Cl}(M)$ ) if and only if  $M = \text{End}(X)$  for some graph  $X$  which is a hull (and in particular, whose core is complete).*

**Proof** Suppose that  $M$  is closed. Then  $M = \text{End}(X)$ , where  $X = \text{Gr}(M)$ ; so  $X = \text{Gr}(\text{End}(X)) = \text{Hull}(X)$ .

Conversely, if  $X = \text{Hull}(X)$ , then  $\text{End}(X) = \text{End}(\text{Gr}(\text{End}(X))) = \text{Cl}(\text{End}(X))$ .