Acyclic orientations of graphs

Celia Glass and Peter Cameron

CSG, 23 March 2012
Thanks to Jo Ellis-Monaghan for the following:

Classical Mathematics--Geometry
The Mathematics of: Measurement, Surveying, Architecture, Astronomy, Volumes and Areas

Static Measurement
(Euclid of Alexandria
325 BC-265 BC)

Mathematics of the Future--Combinatorics
The Mathematics of: Computer Chips, the Internet, Electrical Circuits, Cell Phone Coverage, Transportation Networks, Social Interactions, Genetics, Food Webs….

Interconnections and Relations
(William Tutte 1917-2002)

Early Mathematics--Arithmetic
The Mathematics of: Counting, Money, Inventory, Taxes, Census, Apportionment, Calendars

Enumeration
(Cavemen Prehistory)

www.cartoonstock.com
www.valdostamuseum.org/hamsmith/eghier.html

The Mathematics of: Projectiles, Optimization, Engineering, Machines, Rockets, Planetary Motion, Gravity…

Motion in time and space

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*Interconnections and Relations*
A mobile phone mast
Ofcom manage UK spectrum

- Frequency assignment algorithms for 38Ghz (microwave) bandwidth
- Academic formulation is all-at-once
  - PhD under EPSRC & Ofcom (Ian Davies)
- Assignment done by Ofcom one-at-a-time
  - research contract from Ofcom with Rutherford Appleton Laboratory +
What Is Frequency Assignment?

- *Radio spectrum* is a finite resource
- Prohibiting factor is *interference*
- Basic requirement is for efficient re-use of available spectrum, in different locations
- The spectrum is divided into discrete, uniformly spaced channels - *frequencies*
- Interference between transmitters is measured in *separation distance* between their frequencies

...
Interference Graph Representation

Graph theoretic representation, \( G=(V,E) \)

- Vertex set \( V \) represents the transmitters
- An edge \((u,v) \in E\) occurs between transmitters which potentially may interfere

A Frequency Assignment is a mapping \( f:V \rightarrow F \)

where \( F=\{1,2, \ldots, N\} \) is the set of available frequencies. Separation constraints:

\[
|f(u) - f(v)| \geq d_{uv} \quad \forall (u,v) \in E
\]

Objective = Minimise the number of frequencies required, the span \( Sp(G) \)
Previous solution approaches to FAP

• Several large projects
  • CALMA/RA initiatives/ROADEF, etc.

• Meta-heuristic approaches:
  • Tabu search - successful but only with problem specific tuning
  • SA - successful but requires extensive tuning for each instance type. GA’s - not as successful but requires less tuning than TS/SA
  • Neural Networks - generally not successful
  • Constraint Satisfaction

• Optimisation approaches:
  • branch and bound/polyhedral methods
Problem Comparison

Graph Colouring
- Chromatic Number Problem (CNP)
- k-colouring problem
- Partitioning
- Separation distances all equal to 1
- $G_{1000,0.5}$ a typical data set.
- No a priori structure – hence difficult
- Many good heuristic methods exist

Frequency Assignment
- Minimise no. frequencies/ span
- Minimise interference
- Number Assignment
- Includes separation distances
- FAP instances generally smaller
- Structure exists in the interference graph
Representations of Frequency Assignment

Default representation of solution is
Numbering of vertices - like colouring of vertices gives enormous solution space

Directed Acyclic Graphs (DAG)
captures essences of span, but

how large is DAG space?
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An orientation of $G$ is a digraph obtained by replacing each edge $\{v, w\}$ by one or other of its orderings, i.e. either $(v, w)$ or $(w, v)$. An orientation is acyclic if it does not contain any directed cycles, i.e. distinct vertices $v_0, \ldots, v_{k-1}$ such that $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-2}, v_{k-1}), (v_{k-1}, v_0)$ are arcs.
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Every graph has an acyclic orientation. Number the vertices from 1 to $n$, and orient every edge from the vertex with smaller number to the vertex with the larger.
Acyclic orientations and colourings

Let \( AO(G) \) be the set of acyclic orientations of \( G \), and \( Col(G) \) the set of proper colourings of \( G \) (with an arbitrary number of colours, but the colours are ordered and all of them are used).
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- Given a colouring $c$ of $G$ with $k$ colours, let the colour classes be $C_1, C_2, \ldots, C_k$; number the vertices in $C_1$, then those in $C_2$, and so on up to $C_k$. Now orient edges from smaller number to greater. The result is an acyclic orientation.
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- Given an acyclic orientation $D$ of $G$, let $C_1$ be the set of sources (vertices which are not the target of any arc), $C_2$ the set of sources in the induced subgraph on $V(G) \setminus C_1$, and so on. There are no edges within any $C_i$, so $(C_1, C_2, \ldots, C_k)$ is a proper colouring of $G$, which we call the canonical colouring associated with the acyclic orientation.
Remarks

The “canonical” construction can be applied to any orientation of $G$; it terminates (with $C_1 \cup C_2 \cup \cdots \cup C_k = V(G)$) if and only if the orientation is acyclic. So this is a polynomial-time algorithm to test whether an orientation is acyclic and to find the corresponding colouring if it is.
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The canonical colouring has an alternative description: $C_i$ is the set of all vertices $v$ for which the longest directed path ending at $v$ (in the orientation $D$) contains $i$ vertices. In particular, the number of colours used is equal to the number of vertices in the longest directed path in the orientation: this number is called the span of $D$, written $\text{Span}(D)$. 
How many colours?

At risk of confusion, we define the span of an undirected graph $G$ to be the number of vertices in the longest path of $G$; we denote this also by $\text{Span}(G)$. Also, $\chi(G)$ denotes the chromatic number of $G$, the least number of colours in any proper colouring of $G$. 

Theorem

For any acyclic orientation $D$ of $G$, we have $\chi(G) \leq \text{Span}(D) \leq \text{Span}(G)$.

Question

Is it true that, for every integer $k$ with $\chi(G) \leq k \leq \text{Span}(G)$, there is an acyclic orientation $D$ of $G$ with $\text{Span}(D) = k$?
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**Theorem**

- For any acyclic orientation $D$ of $G$, we have $\chi(G) \leq \text{Span}(D) \leq \text{Span}(G)$.
- There are acyclic orientations $D_1$ and $D_2$ for which $\text{Span}(D_1) = \chi(G)$ and $\text{Span}(D_2) = \text{Span}(G)$.
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Number of directed acyclic graphs on 10 vertices

\[ A_{10} = 3628800x^{45} + 146966400x^{44} \\
+ 2899411200x^{43} + 37126101600x^{42} \\
+ 346868600400x^{41} + 2520365009400x^{40} \\
+ 14823549568800x^{39} + 72525982284000x^{38} \\
+ 30105630457600x^{37} + 1076055091414800x^{36} \\
+ 3349674724515840x^{35} + 9163072757462400x^{34} \\
+ 22184317673849520x^{33} + 47807980082864190x^{32} \\
+ 92129542599754800x^{31} + 159344586974784960x^{30} \\
+ 248071275833167080x^{29} + 348409073759608260x^{28} \\
+ 442176547815875040x^{27} + 507675000725890200x^{26} \\
+ 527641018776771732x^{25} + 496515058907266500x^{24} \\
+ 422913488921810640x^{23} + 325827430873816320x^{22} \\
+ 226797475663517760x^{21} + 142397107185335940x^{20} \\
+ 80476050938371200x^{19} + 40832558916877560x^{18} \\
+ 18542265211960110x^{17} + 7508190221370540x^{16} \\
+ 2699438041234560x^{15} + 857577282883200x^{14} \\
+ 239434790091840x^{13} + 58405018216860x^{12} \\
+ 12368745491760x^{11} + 2259242749800x^{10} \\
+ 353530511420x^9 + 47056700160x^8 \\
+ 5284309680x^7 + 495329520x^6 \\
+ 38167920x^5 + 2362500x^4 \\
+ 113280x^3 + 3960x^2 \\
+ 90x + 1 \]
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Trivially, the number is between 1 and \( \min\{n!, 2^m\} \). Next week, Robert Schumacher will say more about some aspects of this question, and will give substantial improvements to this trivial bound.
The Petersen graph $G$ has $\chi(G) = 3$ and Span($G$) = 10. The numbers of acyclic orientations of $G$ with span equal to 3, 4, ..., 10 are 80, 640, 2160, 4920, 4080, 2880, 1680, 240 respectively. There are 16680 acyclic orientations, falling into 168 isomorphism classes.
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The Petersen graph, continued

The chromatic polynomial of the Petersen graph is

\[
P(q) = q(q - 1)(q - 2) \times (q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352).
\]

We find that \( P(3) = 120 \), whereas there are only 80 acyclic orientations with span 3.
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We find that \( P(3) = 120 \), whereas there are only 80 acyclic orientations with span 3.
So, in general, the composition of the two mappings is not the identity.
The vertices of the Petersen graph can be identified with the 2-element subsets of the set $\{1, 2, 3, 4, 5\}$, two vertices adjacent if and only if they are disjoint. The colour classes in a typical 3-colouring are

$\{12, 13, 14, 15\}, \{23, 24, 25\}, \{34, 35, 45\}$. 
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\]

There are 20 such partitions (the number of choices of the numbers taking the place of 1 and 2), and \( 3! = 6 \) orderings of the colours. Now we see that, in the colouring with the colour classes (in order)

\[
\{12, 23, 25\}, \{14, 24, 34, 45\}, \{13, 15, 35\},
\]

the vertex 24 is a sink in the corresponding acyclic orientation, so it is transferred to the first class in the canonical colouring.
Proof of the theorem

The fact that $\text{Span}(D)$ lies between $\chi(G)$ and $\text{Span}(G)$ is clear.
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- What can be said about the distribution of the numbers of acyclic orientations with each possible span?
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If we take a path containing $\text{Span}(G)$ vertices, direct it consistently, and extend arbitrarily to an acyclic orientation $D$ of $G$, the result will have $\text{Span}(D) = \text{Span}(G)$.

Question

- What can be said about the distribution of the numbers of acyclic orientations with each possible span?

- Is it true that every colouring of $G$ with $\chi(G)$ colours is, up to re-ordering of the colours, the canonical colouring associated with some acyclic orientation?
A recurrence relation for the number $a(n, m)$ of acyclic digraphs on the vertex set $\{1, \ldots, n\}$ with $m$ arcs was found by Bender, Richmond, Robinson and Wormald in 1986:
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**Theorem**

Let $A_n(x) = \sum_m a(n, m)x^m$. Then

$$A_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (1 + x)^{i(n-i)} A_{n-i}(x).$$
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$$A_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (1 + x)^{i(n-i)} A_{n-i}(x).$$

Since the number of graphs with on $\{1, \ldots, n\}$ with $m$ edges is simply $\binom{n(n-1)/2}{m}$, we can easily compute (for given $n,m$) the average number of acyclic orientations of such a graph.
\[ n = 10 \]

\[ A_{10} = 3628800 x^{45} + 146966400 x^{44} + 289941200 x^{43} + 37126101600 x^{42} + 34686600400 x^{41} + 2520365009400 x^{40} + 14823549568800 x^{39} + 72525982284000 x^{38} + 30105630457600 x^{37} + 1076055091414800 x^{36} + 3349674724515840 x^{35} + 9163072757462400 x^{34} + 22184317673849520 x^{33} + 47807980082864190 x^{32} + 92129542599754800 x^{31} + 159344589748784960 x^{30} + 248071275833167080 x^{29} + 348409073759608260 x^{28} + 442176547815875040 x^{27} + 507675000725890200 x^{26} + 527641018776771732 x^{25} + 496515058907266500 x^{24} + 422913488921810640 x^{23} + 325827430873816320 x^{22} + 226797475663517760 x^{21} + 142397107185335940 x^{20} + 8047605093871200 x^{19} + 40832558916877560 x^{18} + 18542265211960110 x^{17} + 7508190221370540 x^{16} + 2699438041234560 x^{15} + 857577282883200 x^{14} + 239434790091840 x^{13} + 58405018216860 x^{12} + 12368745491760 x^{11} + 2259242749800 x^{10} + 353530511420 x^9 + 47056700160 x^8 + 5284309680 x^7 + 495329520 x^6 + 38167920 x^5 + 2362500 x^4 + 113280 x^3 + 3960 x^2 + 90 x + 1 \]
Average number of acyclic orientations, $n = 10$

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Proportion of orientations which are acyclic, $n = 10$

The axes give number $m$ of edges (0 to 45) and average number of acyclic orientations divided by $2^m$ (1 to $10!/2^{45}$).
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Given a graph $G$ on $n$ vertices, how many acyclic orientations does $G$ have? Stanley proved:
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**Theorem**

The number of acyclic orientations of an $n$-vertex graph $G$ is $(-1)^n P_G(-1)$, where $P_G(q)$ is the chromatic polynomial of the graph $G$. 

Now this is an evaluation of the Tutte polynomial of $G$, so it follows from the results of Jaeger, Vertigan and Welsh that the problem of counting the acyclic orientations of a general graph is #P-complete.
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Goldberg and Jerrum, and others, have investigated approximate evaluations of the Tutte polynomial (in the sense of the existence of a fully polynomial randomized approximation scheme, or FPRAS). As far as we are aware, the question has not been settled for this particular evaluation.
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Cameron, Jackson and Rudd defined an orbital chromatic polynomial (depending on a graph $G$ and a group of automorphisms of $G$) whose value at a positive integer $q$ is the number of orbits of the group on proper $q$-colourings of $G$. Substituting $q = -1$ in the polynomial does not, in general, give the number of orbits on acyclic orientations of $G$. However, there is a twisted orbital chromatic polynomial which does have this property. I might tell you about this some other time!
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The arc-flip graph

The arc-flip graph of $G$ is the graph whose vertex set is the set $AO(G)$ of acyclic orientations of $G$, with two orientations $D_1$ and $D_2$ joined if they differ in the direction of a single arc.
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The arc-flip graph of $G$ is the graph whose vertex set is the set $\text{AO}(G)$ of acyclic orientations of $G$, with two orientations $D_1$ and $D_2$ joined if they differ in the direction of a single arc. The distance between two elements $D_1, D_2 \in \text{AO}(G)$ is the number of arcs whose direction is different in $D_1$ and $D_2$. 

**Theorem**

The distance between $D_1$ and $D_2$ is equal to the distance between the corresponding vertices of the arc-flip graph. In other words, it is possible to move from $D_1$ to $D_2$ by successively reversing the direction of arcs, so that each arc is reversed at most once, and we stay inside the set $\text{AO}(G)$ of acyclic orientations at every step.

**Corollary**

The diameter of the arc-flip graph of $G$ is equal to the number $m$ of edges of $G$; and the unique acyclic orientation $D^*$ at maximum distance from $D_1$ is obtained by reversing all the edges.
The arc-flip graph of $G$ is the graph whose vertex set is the set $AO(G)$ of acyclic orientations of $G$, with two orientations $D_1$ and $D_2$ joined if they differ in the direction of a single arc. The distance between two elements $D_1, D_2$ pf $AO(G)$ is the number of arcs whose direction is different in $D_1$ and $D_2$.

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Corollary

The diameter of the arc-flip graph of $G$ is equal to the number $m$ of edges of $G$; and the unique acyclic orientation $D^*$ at maximum distance from $D$ is obtained by reversing all the edges.
The previous result means, in particular, that in every acyclic digraph, there is at least one arc which can be reversed without destroying the acyclicity.
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Another remark is that an acyclic digraph \( D \) and its reversal \( D^* \) have the same span, though the canonical colourings are not always the same up to reversal of the order of colours.
A Markov chain

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The state space is the set $\text{AO}(G)$ of acyclic orientations of $G$. A move consists of the following:

- Choose a random arc $e$ (uniformly).
- If $e$ is reversible (that is, if the digraph $D'$ obtained by reversing $e$ is acyclic), then reverse it; otherwise do nothing.
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The Markov chain is connected, and aperiodic (if $G$ is not a tree, there is at least some move which is forbidden), and is symmetric (the probabilities of moving from $D$ to $D'$ and from $D'$ to $D$ are equal); so the limiting distribution is uniform.
The big question, of course, is:

**Question**

*What is the mixing time of this Markov chain?*

This may be a hard question. For, if the Markov chain is rapidly mixing, then we have an efficient way to sample acyclic orientations of $G$ uniformly, and hence there will be an FPRAS for the number of acyclic orientations. We have some partial results.
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However, the moves may be quite efficient in practical. For example, with the Petersen graph, the distribution of spans of the acyclic orientations is reproduced remarkably accurately after runs of just 20 steps (only a little more than the number of edges).
Another Markov chain

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