Solutions to Exercises
Chapter 11: Graphs

1 There are 34 non-isomorphic graphs on 5 vertices (compare Exercise 6 of Chapter 2). How many of these are (a) connected, (b) forests, (c) trees, (d) Eulerian, (e) Hamiltonian, (f) bipartite?

The numbers are (a) 21; (b) 10; (c) 3; (d) 4; (e) 8; (f)13. (The answer in the book for (a) is wrong; there are 13 *disconnected* graphs.) It is probably simplest to list all 34 graphs and check the six properties.

For jobs of this kind the *Atlas of Graphs* (ed. R. C. Read and R. J. Wilson), Oxford University Press, 1998, is useful.

2 Show that the Petersen graph (Section 11.12) is not Hamiltonian, but does have a Hamiltonian path.

In the drawing of the Petersen graph in Figure 11.4 on page 183, we distinguish five edges of the *outer pentagon*, five edges of the *inner pentagram*, and five *crossing edges*.

A Hamiltonian circuit must return to its starting point, and so must use an even number (2 or 4) of crossing edges.

If there are two crossing edges, then their ends in both the outer pentagon and the inner pentagram must be joined by paths of length 4. This implies that these ends must be adjacent in both the outer pentagon and the inner pentagram, which is impossible.

If there are four crossing edges, then we must have three edges in both the outer pentagon and the inner pentagram, including two through each of the points not on chosen crossing edges. There is a unique such configuration, but it consists of two 5-cycles rather than a Hamiltonian circuit.

A Hamiltonian path is easily found: follow a path containing four edges of the outer pentagon, then take a crossing edge, then follow four edges of the inner pentagram.

3 Show that the greedy algorithm does not always succeed in finding the path of least weight between two given vertices in a connected edge-weighted graph.

It is not completely clear what is meant by the 'greedy algorithm' in this case. If we have to construct a path from x to y, we take the algorithm as follows: start at x, and at any stage, take that edge of smallest weight whose other end is nearer to y than the current vertex. If x and y are joined by two paths of length 2, with edge weights 3 and 3 on one path, 4 and 1 on the other, then this algorithm will fail. (For this example, it would succeed in finding a shortest path from y to x. If, however, we take x and y to be joined by two paths of length 3, with edge weights $2,1,2$ on one and $1,4,1$ on the other, then it fails in both directions.)

4 Consider the modification of the greedy algorithm for minimal connector. Choose the edge e for which $w(e)$ is minimal subject to the conditions that $S+e$ contains no cycle *and e shares a vertex with some previously chosen edge (unless $S=\emptyset$)*. Prove that the modified algorithm still correctly finds a minimal connector.

We suppose that all the edge weights are different. Then there is a unique minimal spanning tree T, found by GA (the greedy algorithm, as described in the text). If we apply MGA (the modified greedy algorithm, as in the question) to T, ignoring all other edges, it must choose all the edges of T in some order, say $e_1, e_2, \ldots, e_{n-1}$.

Suppose that, when applied to the whole graph G, the MGA doesn't choose T. Suppose i is minimal with respect to the property that it doesn't choose e_i, but chooses another edge f instead. (Note that $i > 1$, since e_1 is the edge of smallest weight.) The edges $\{e_1, \ldots, e_{i-1}\}$ form a subtree T', and f is chosen as the edge of smallest weight joining T' to a new vertex. There is a unique circuit in $T \cup \{f\}$, containing a unique edge (other than f) joining T' to a vertex outside it, say e_j. Then $w(f) < w(e_j)$. But if we delete e_j from T and replace it with f, we obtain a tree of smaller weight, contradicting the minimality of T.

If we allow some edge weights to vary continuously so that they become equal to other edge weights, both the weight of a minimal connector and the weight produced by MGA vary continuously, and so they remain equal in this case.

5 Let $G=(V,E)$ be a multigraph in which every vertex has even valency. Show that it is possible to direct the edges of G (that is, replace each unordered pair $\{x,y\}$ by the ordered pair (x,y) or (y,x)) so that the in-valency of any vertex is equal to its out-valency.

By (11.4.1), each connected component of G is Eulerian. Take a closed Eulerian trail in each component, and direct the edges according to their direction of transit while following this trail. Now we must leave each vertex as often as we enter it; so the in- and out-valency of any vertex in the directed graph are equal.

6 Let G be a graph on n vertices. Suppose that, for all non-adjacent pairs x, y of vertices, the sum of the valencies of x and y is at least $n-1$. Prove that G is connected.

Let x and y be two vertices. If they are adjacent, there is a path of length'1 between them; so suppose not. Let X and Y be the sets of neighbours of x and y respectively. Then $|X|+|Y| \geq n-1$. But $X \cup Y \subseteq V(G) \setminus \{x, y\}$; so $|X \cup Y| \leq n-2$. Hence $|X \cap Y| \geq 1$, and there is a path of length 2 from x to y (via a vertex in this intersection). So G is connected (and has diameter at most 2).

Remark. This is best possible — for n even, the disjoint union of two complete graphs of size $n/2$ demonstrates this. Note that strengthening the bound by just one forces the graph to be Hamiltonian (Ore's Theorem).

7 (a) Prove that a connected bipartite graph has a unique bipartition.
(b) Prove that a graph G is bipartite if and only if every circuit in G has even length.

(a) If G is connected, then two points lie in the same bipartite block if and only if the length of a path joining them is even. This determines the bipartition uniquely.

(b) Since a graph is bipartite if and only if each connected component is bipartite, it is enough to prove this for a connected graph. Now if a connected bipartite graph contains an odd circuit, then a vertex on this circuit is joined to itself by a path of odd length, and so lies in the opposite bipartite block to itself, a contradiction. Conversely, suppose that G is connected, and that all its circuits have even length. Then, for any vertices x and y, the parity of all paths joining x and y is the same (else an odd circuit would be formed). So, given a vertex x, we may divide the graph into the sets of vertices joined to x by even, resp. odd, paths. Any edge must join vertices in different sets, so this is a bipartition.

8 Choose ten towns in your country. Find from an atlas (or estimate) the distances between all pairs of towns. Then

(a) find a minimal connector;

(b) use the 'twice-round-the-tree' algorithm to find a route for the Travelling Salesman.

How does your route in (b) compare with the shortest possible route?

The answer clearly depends on the towns chosen.

The most certain way to establish the shortest possible route for the Travelling Salesman is by exhaustive search. Keeping the first town fixed, permute the remaining nine in all possible ways (using the algorithm of (3.12.4)); for each permutation, calculate the length of the circuit, and record the shortest length. A computer will perform this calculation in a few seconds.

9 Consider the result of Chapter 6, Exercise 7, viz.

> *Let $\mathcal{F} = (A_1, \ldots, A_n)$ be a family of sets having the property that $|A(J)| \geq |J| - d$ for all $J \subseteq \{1, \ldots, n\}$, where d is a fixed positive integer. Then there is a subfamily containing all but d of the sets of $\mathcal{F}$, which possesses a SDR.*

Prove this by modifying the proof of Hall's Theorem from König's given in the text.

Following the proof of (11.10.3), we see that an edge-cover S satisfies $|S| \geq n - d$; by König's Theorem, there is a matching of size (at least) $n - d$, that is, a SDR for all but d of the sets.

For the converse, assume that the version of Hall's Theorem given in this question is valid, and let G be a bipartite graph with bipartite blocks of sizes m and n, and with smallest edge-cover of size k. Clearly there cannot be a matching of size larger than k; we have to find one of size k. Let the vertices in the two bipartite blocks be $X = \{v_1, \ldots, v_m\}$ and $Y = \{w_1, \ldots, w_n\}$, and let A_i be the set of neighbours of w_i. Given $J \subseteq \{1, \ldots, n\}$, let $W(J) = \{w_j : j \in J\}$. Then $A(J) \cup (Y \setminus W(J))$ is an edge-cover; and so

$$|A(J)| + n - |J| \geq k,$$

whence $|A(J)| \geq |J| - d$, where $d = n - k$.

By Hall's Theorem, there is a set J, with $|J| = n - d$, for which the family $(A_j : j \in J)$ has an SDR, say $(x_j : j \in J)$. Then the edges $\{x_j, y_j\}$, for $j \in J$, form a matching, of size $n - d = k$.

10 König's Theorem is often stated as follows:

> *The minimum number of lines (rows or columns) which contain all the non-zero entries of a matrix A is equal to the maximum number of independent non-zero entries,*

where a set of matrix entries is *independent* if no two are in the same row or column. Show the equivalence of this form with the one given in the text.

Assume König's Theorem in the 'bipartite graph' form, and let A be a matrix. Form a graph as described in the question. Now a non-zero matrix entry corresponds to an edge of the graph, and a set of independent such entries to a set of disjoint edges, that is, a matching. Also, a set of lines containing all non-zero entries translates to a set of vertices incident with all edges, that is, an edge-cover. So the 'matrix' form of the theorem follows.

Conversely, assume the 'matrix' form, and let G be a bipartite graph with bipartite blocks $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$. Let A be the $m \times n$ matrix with (i, j) entry 1 if $\{v_i, w_j\}$ is an edge of G, and 0 otherwise. Now the translation reverses: a matching in G is a set of independent non-zero entries of A, and an edge-cover is a set of lines containing all non-zero entries. So the 'bipartite graph' form follows.

11 In this exercise, we translate the 'stepwise improvement' algorithm in the proof of the Max-Flow Min-Cut Theorem into an algorithm for König's Theorem.

Let G be a bipartite graph with bipartition $\{A,B\}$. We observed in the text that an integer-valued flow f in $N(G)$ corresponds to a matching M in G, consisting of those edges $\{a,b\}$ for which the flow in (a,b) is equal to 1. Now consider the algorithm in the proof of the Max-Flow Min-Cut Theorem, which either increases the value of the flow by 1, or finds a cut. Suppose that we are in the first case, where there is a path

$$(s,a_1,b_1,a_2,b_2,\ldots,a_r,b_r,t)$$

in the underlying graph of $N(G)$ along which the flow can be increased. Then

> $(a_1,b_1,\ldots,a_r,b_r)$ *is a path in* G, *such that all the edges* $\{b_i,a_{i+1}\}$ *but none of the edges* $\{a_i,b_i\}$ *belong to* M; *moreover, no edge containing* a_1 *or* b_r *is in* M.

Such a path in G is called an *alternating path* with respect to M. (An alternating path starts and ends with an edge not in M, and edges not in and in M alternate. Moreover, since no edge of M contains a_1 or b_r, it cannot be extended to a longer such path.)

Show that, if we delete the edges $\{b_i,a_{i+1}\}$ from M ($i=1,\ldots,r-1$), and include the edges $\{a_i,b_i\}$ ($i=1,\ldots,r$), then a new matching M' with $|M'|=|M|+1$ is obtained.

So the algorithm is:

> WHILE there is an alternating path, apply the above replacement to find a larger matching.
>
> When no alternating path exists, the matching is maximal.

The question doesn't actually ask you to do anything! But here is a set-up which both allows efficient search for an alternating path, and also allows a simple proof from König's Theorem that the matching is maximum if no alternating path exists.

Let G be a bipartite graph with bipartition $\{A,B\}$, and let $M=\{\{a_1,b_1\},\ldots,\{a_r,b_r\}\}$ be a matching in G. We form an auxiliary digraph $A(M)$ as follows. The vertices of $A(M)$ are symbols $e_1,\ldots,e_r$ corresponding to the edges of M, and two new symbols s and t. The edges in $A(M)$ are as follows:

- (e_i, e_j), if there is an edge $\{a_i, b_{i+1}\}$ in G;
- (s, e_i), if there is an edge $\{a, b_i\}$ in G, where a is on no edge of M;
- (e_j, t), if there is an edge $\{a_j, b\}$ in G, where b is on no edge of M.

Now, if there is a directed path from s to t in $A(M)$, then it arises from an alternating path in G, and so a larger matching exists.

Suppose that no directed path from s to t exists in $A(M)$. Let S be the set of vertices reachable from s in $A(M)$, and T the remaining vertices. Then, if $e_i \in S$ and $e_j \in T$, then there is no edge from a_i to b_j, or from a to b_j with a not covered by M, or from a_i to b with b not covered by M. It follows that

$$\{b_i : e_i \in S\} \cup \{a_j : e_j \in T\}$$

is an edge-cover of G. Its cardinality is equal to that of M; hence no larger matching can exist.

12 Let G be a graph with adjacency matrix A. Prove that the (i,j) entry of A^d is equal to the number of walks of length d from i to j.

The (i,j) entry of A^d is equal to

$$\sum_{k_1,\ldots,k_{d-1}} a_{ik_1} a_{k_1k_2} \cdots a_{k_{d-1}j}.$$

A term in this sum is zero unless $\{i,k_1\}$, $\{k_1,k_2\}$, …, $\{k_{d-1}, j\}$ are all edges (that is,

$$(i, k_1, k_2, \ldots, k_{d-1}, j)$$

is a walk of length d from i to j), in which case it is 1. So the sum counts the number of walks of length d from i to j.

13 This exercise proves the 'friendship theorem': *in a finite society in which any two members have a unique common friend, there is somebody who is everyone else's friend.* In graph-theoretic terms, a graph on n vertices in which any two vertices have exactly one common neighbour, possesses a vertex of valency $n-1$, and is a 'windmill'.

Step 1 Let the vertices be $1,\ldots,n$, and let A_i be the set of neighbours of i. Using the de Bruijn–Erdős Theorem (Chapter 7), or directly, show that *either* there is a vertex of valency $n-1$, *or* all sets A_i have the same size (and the graph is regular). In the latter case, the sets A_i are the lines of a projective plane (Chapter 9).

Step 2 Suppose that G is regular, with valency k. Use the eigenvalue technique of Section 11.11 to prove that $k=2$.

(a) Using the de Bruijn–Erdős Theorem: Let A_i be the set of neighbours of j. By assumption, $|A_j \cap A_k| = 1$ for all $j \neq k$. Hence either all the sets A_j have the same cardinality k (so that the graph is regular) or, say $|A_1| = 1$ and $|A_j| = 2$ for $j \neq 1$. In the latter case, vertex 1 is joined to all others, and the remaining vertices pair off as in the figure.

Directly: We show that two non-adjacent vertices x and y have the same valency. There is one vertex z joined to x and y; any other neighbour of x is joined to a unique neighbour of y, and *vice versa*.

Suppose that G is not regular. Any two vertices with different valency are joined. If A is the set of all vertices with valency a, then either A contains just one vertex, or two vertices in A have $n-|A|$ common neighbours at least, so that A contains all but one vertex. For $n>2$, we have $2(n-1)>n$, and so there must be a singleton set $A=\{v\}$. Now v is joined to all other vertices, and we conclude as above that the graph is a windmill.

(b) Let G be regular with valency k. Then the adjacency matrix A satisfies $A^2 = kI + (J-I) = (k-1)I + J$, where J is the all-1 matrix (by Question 12). Now the all-1 vector j is an eigenvector with eigenvalue k. Since A is symmetric, any other eigenvector w is orthogonal to j, whence $Jw=0$, and so $A^2w = (k-1)w$. So the other eigenvalues of A are $\pm\sqrt{k-1}$. Let their multiplicities be f and g. The sum of the eigenvalues is the trace of A, which is zero. SO

$$k + (f-g)\sqrt{k-1} = 0.$$

Thus $\sqrt{k-1}$ is rational, and hence integral: that is, $k = u^2+1$ for some positive integer u, and we have

$$f - g = -(u^2+1)/u.$$

This implies that u divides 1; so $u = 1$, $k = 2$, $n = 3$, and the graph is a windmill with one sail.

14 The 'Trackwords' puzzle in the *Radio Times* consists of nine letters arranged in a 3×3 array. It is possible to form an English word from all nine letters, where consecutive letters are adjacent horizontally, vertically or diagonally. Consider the problem of setting the puzzle; more specifically, of deciding in how many ways a given word (with all its letters distinct) can be written into the array.

(a) Formulate the problem in graph-theoretic terminology.

(b) (COMPUTER PROJECT.) In how many ways can it be done?

The question asks for the number of Hamiltonian paths in a 9-vertex graph.

This number can be determined by trying all permutations of the order of the nine vertices to see which are Hamiltonian paths, which can be calculated easily (using the algorithm of (3.12.4) to generate the permutations).

15 The following algorithm, due to Peter Johnson, makes the proof of Ore's theorem (11.5.1) constructive. Verify the details.

Let G satisfy the hypotheses of Ore's theorem. Choose any circuit C on the vertex set of G. If $E(C) \subseteq E(G)$, we are done. Otherwise, let $d = |E(C) \setminus E(G)|$; we find a new circuit C' with $|E(C') \setminus E(G)| < d$. A finite number of such steps finds the required Hamiltonian circuit.

Take any edge $e \in E(C) \setminus E(G)$. The graph G' with edge set $E(G) \cup E(C) \setminus \{e\}$ satisfies Ore's condition and has a Hamiltonian path $E(C) \setminus \{e\}$. Now proceed as in the proof of (11.5.1) to find a Hamiltonian circuit C' in G'.

15. Follow the details as given. The Hamiltonian circuit obtained in G' does not contain the edge e, so it has fewer edges not in G than C. After a finite number of steps, no edges not in G are used.