Problems from the Nineteenth British Combinatorial Conference

Edited by Peter J. Cameron
School of Mathematical Sciences
Queen Mary, University of London
London E1 4NS
U.K.
p.j.cameron@qmul.ac.uk

August 2003

Abstract

Most of these problems were presented at the problem session of the 19th British Combinatorial Conference at the University College of North Wales, Bangor, 29 June–4 July 2001. I have added some problems given to me after the session. As previously, the problems are circularly ordered; you may start at any point.

Problem 1 (BCC19.1) Tight single-change covering designs. Proposed by D. A. Preece.

Correspondent: D. A. Preece
School of Mathematical Sciences, Queen Mary,
University of London, London E1 4NS, U.K.
d.a.preece@qmul.ac.uk

The array

1	1	1	1	1	3	3	3	2	2
2	2	5	6	7	7	7	7	7	6
3	4	4	4	4	4	6	5	5	5

is a *tight single-change covering design* as

(a) each pair of its entries occurs in at least one column (block), so all pairs are "covered";

(b) when each block is formed from the previous one, only a single change is made;

(c) when each new block is formed, the new entry occurs only with entries with which it has not previously been paired, so the arrangement is "tight".

Tight single-change covering designs with block sizes 2 (trivial), 3, 4 and 5 are known and in the literature.

Problem: Find tight single-change covering designs with block size greater than 5.

Problem 2 (BCC19.2) Three problems on partial Latin squares. Proposed by R. Bean.

Correspondent: R. Bean
IPM, PO Box 19395-5746, Tehran, Iran
rwb@ipm.ir

1. If a partial Latin square P of order $n \geq 6$ has fewer than $\lfloor n^2/4 \rfloor$ entries and is completable to a Latin square, then there is an *intercalate* I (a 2×2 Latin square) such that $P \cup I$ is completable.

2. If a partial Latin square P of order n has fewer than $\lfloor n^2/4 \rfloor$ entries, then it has an even number of completions to a Latin square.

3. The size $\mathrm{lcs}(n)$ of the largest critical set in a Latin square of order n satisfies

$$\max\{A(n), B(n), C(n), D(n)\} \leq \mathrm{lcs}(n) \leq E(n),$$

where

$$\begin{aligned}
A(n) &= (n^2 - n)/2, \\
B(n) &= \max\{(3(pq)^2 - pq^2 - qp^2 - pq)/4 : p, q \in \mathbb{N}, pq = n\}, \\
C(n) &= |\{(i,j) : 0 \leq i, j < n, i \,\&\, j \neq 0\}|, \\
D(n) &= \max\{\mathrm{lcs}(p)q^2 + \mathrm{lcs}(q)p^2 - \mathrm{lcs}(p)\,\mathrm{lcs}(q) : p, q \in \mathbb{N}, pq = n\}, \\
E(n) &= n^2 - n^{\log 3/\log 2}.
\end{aligned}$$

Note that $B(n) \leq D(n)$ for all n; the reason for including B is in case the bound D proves to be incorrect. The function & in the definition of $C(n)$ is "bitwise and": write the arguments i and j in base 2, then multiply the digits in each position and interpret the result as the base 2 representation of $i \,\&\, j$. The conjectured lower bound $C(n)$ is Sequence A080572 in the *Online Encyclopaedia of Integer Sequences* [8]; see also Gower [5].

Problem 3 (BCC19.3) Designs with three eigenvalues. Proposed by R. A. Bailey.

Correspondent: R. A. Bailey
School of Mathematical Sciences, Queen Mary,
University of London, London E1 4NS, U.K.
r.a.bailey@qmul.ac.uk

Let D be a connected binary equireplicate proper incomplete-block design with v treatments. (This means that no treatment occurs more than once in a block; each block contains a fixed number k of treatments, with $k < v$; each treatment occurs in a fixed number r of blocks; and the Levi graph is connected.) The *concurrence matrix* is the $v \times v$ matrix Λ with rows and columns indexed by treatments, whose (i, j) entry is the number of blocks in which i and j occur.

If D is balanced, then Λ has two distinct eigenvalues. The problem is, when does Λ have just three distinct eigenvalues? This occurs at least in the following cases:

- duals of non-symmetric BIBDs;
- partially balanced IBDs with two associate classes;
- duals of symmetric partially balanced IBDs with two associate classes;
- duals of non-symmetric partially balanced IBDs with two associate classes and having an efficiency factor 1 (if not balanced).

Are there any others?

Problem 4 (BCC19.4) Forced domination number of products of cycles. Proposed by E. S. Mahmoodian.

Correspondent: E. S. Mahmoodian
Department of Mathematical Sciences, Sharif University
of Technology, PO Box 11365-9415, Tehran, Iran
emahmood@sharif.edu

A *dominating set* in a graph G is a set S of vertices such that every vertex not in S is adjacent to a vertex in S. A *forced dominating set* is a set of vertices which is contained in a unique dominating set of minimum cardinality. See [3].

Conjecture: If G is the Cartesian product of k copies of C_{2k+1}, then the smallest forced dominating set in G has cardinality k.

This is known to be true if $k = 2$ or $k = 3$. A prize of 1000000 Rials is offered.

Problem 5 (BCC19.5) Integrated colouring by greedy algorithm. Proposed by R. Cowen.

Correspondent: R. Cowen
Queens College, CUNY, Flushing, NY 11367, U.S.A.
rcowen@nyc.rr.com

A colouring of the vertices of a graph with two colours is said to be *integrated* if at least half of the neighbouring vertices of any vertex v have the opposite colour to v.

It is easy to see that every graph has an integrated colouring, since changing the color of any vertex which fails the condition increases the number of edges whose end vertices have opposite colours. Indeed, this procedure of flipping vertices which aren't integrated must find an integrated colouring in $O(n^2)$ steps, where n is the number of vertices. It is natural to ask if this bound can be lowered by using a particular method of choosing the vertices to flip.

Conjecture: If we perform the above algorithm greedily, choosing at each stage the vertex v such that the excess of the number of same-coloured neighbours of v over the opposite-coloured neighbours is maximum, then the algorithm takes $O(n)$ steps. We allow any initial colouring of the vertices.

Remarks: 1. There has been some discussion of this problem in [10, pp. 616–617].

2. G. Ziegler suggested that a similar problem was posed by P. Erdős [4].

3. For a generalization of the integrated colouring result to more colours see [2].

Problem 6 (BCC19.6) Cospectral graphs. Proposed by R. Häggkvist.

Correspondent: R. Häggkvist
Matematiska Institutionen, Umeå University, Umeå,
S-901 87 Sweden
roland.haggkvist@math.umu.se

Does almost every graph (a proportion tending to 1 as $n \to \infty$) have a cospectral mate (another graph with the same characteristic polynomial?)

The same question can be asked for other graph polynomials such as the chromatic polynomial.

Editor's Note: A recent paper by Haemers and Spence [6] suggests that the answer to the problem may be negative. It is shown that the proportion of graphs having a cospectral mate increases for $n \leq 10$ but is smaller for $n = 11$ than for $n = 10$.

Problem 7 (BCC19.7) An arithmetic graph. Proposed by N. Lichiardopol.

Correspondent: N. Lichiardopol
Université de Nice Sophia Antipolis, Laboratoire 13S Le
Verger 10, Moulin de Redon, Auriol, 13390 France
lichiar@club-internet.fr

Let m be an integer satisfying $m \geq 2$. Let G_m be the graph whose vertex set is the set $\mathbb{Z}$ of integers and whose edges are the pairs $\{x, y\}$ such that $y = x + m$ or $y = x - m$ or $y = xm$ or $y = x/m$.

Prove or disprove:

(a) G_m has a one-way Hamiltonian path (a sequence $(x_i : i \geq 0)$ containing each vertex exactly once such that $\{x_i, x_{i+1}\}$ is an edge for all $i \geq 0$);

(b) G_m has a two-way Hamiltonian path (a sequence $(x_i : i \in \mathbb{Z})$ containing each vertex exactly once such that $\{x_i, x_{i+1}\}$ is an edge for all $i \in \mathbb{Z}$);

(c) G_m can be decomposed into (finite) vertex-disjoint cycles.

Problem 8 (BCC19.8) Directed paths in the cube. Proposed by B. Bollobás and I. Leader.

Correspondent: I. Leader
Department of Pure Mathematics and Mathematical
Statistics, Wilberforce Rd, Cambridge CB3 0WB, U.K.
i.leader@dpmms.cam.ac.uk

Let Q_n denote the n-cube $\{0,1\}^n$. Bollobás and Leader [1] showed that, if A and B are disjoint subsets of Q_n with $|A| = |B| = 2^k$, then there exist $2^k(n-k)$ edge-disjoint paths between A and B. (This is clearly best possible, since if A is a subcube, then there are just $2^k(n-k)$ edges from A to its complement.)

Problem: Regarding Q_n as a poset, suppose that A is a down-set and B is an up-set. Can we find $2^k(n-k)$ paths directed upwards from A to B?

Problem 9 (BCC19.9) Primitive lambda-roots. Proposed by P. J. Cameron.

Correspondent: P. J. Cameron
School of Mathematical Sciences, Queen Mary,
University of London, London E1 4NS, U.K.
p.j.cameron@qmul.ac.uk

A *primitive lambda-root* (PLR) of n is an element of the group of units modulo n whose order is as large as possible. (This maximum order is *Carmichael's lambda-function* $\lambda(n)$.) Let $F(n)$ be the number of PLRs of n.

If primitive roots of n exist (that is, if n is an odd prime power, or twice an odd prime power, or 4), then $F(n) = \phi(\phi(n))$, where ϕ is Euler's function. In general, $F(n) \geq \phi(\phi(n))$. However, as Donald Preece pointed out, equality holds for many other values of n (indeed, for 5 309 906 numbers up to 10^7).

Problem: Does the proportion of positive integers $n \leq x$ which satisfy $F(n) = \phi(\phi(n))$ tend to a limit as $x \to \infty$? If so, what is this limit?

Problem 10 (BCC19.10) Sum-free sets in the square. Proposed by Harut Aydinian.

Correspondent: P. J. Cameron
School of Mathematical Sciences, Queen Mary,
University of London, London E1 4NS, U.K.
p.j.cameron@qmul.ac.uk

This problem was communicated to me by Oriol Serra.

What is the size of the largest sum-free set (one not containing two points and their vector sum) in the square $\{1,\ldots,n\}\times\{1,\ldots,n\}$? In particular, show that the number is $cn^2+O(n)$, and find the constant c.

Upper and lower bounds for c are $\mathrm{e}^{-1/2}=0.6065\ldots$ and $3/5=0.6$ respectively.

Problem 11 (BCC19.11) The second largest maximal *k*-arc. Proposed by J. W. P. Hirschfeld.

Correspondent: J. W. P. Hirschfeld
School of Mathematical Sciences, University of Sussex,
Falmer, Brighton BN1 9QH, U.K.
jwph@susx.ac.uk

Let $\mathrm{PG}(2,q)$ be the Desarguesian projective plane of *square* order q. Prove that $m'(2,q)=q-\sqrt{q}+1$ for $q>9$.

Notes: A k-arc is a set of k points with no three collinear, and $m'(2,q)$ is the size of the second largest maximal k-arc. The size $m(2,q)$ of the largest k-arc is known to be $q+1$ if q is odd, and $q+2$ if q is even. It is known that $m'(2,q)=q-\sqrt{q}+1$ if q is an even power of 2; the problem is to prove this for odd prime power squares.

Problem 12 (BCC19.12) Covering radius of $\mathrm{PGL}(2,q)$. Proposed by P. J. Cameron.

Correspondent: P. J. Cameron
School of Mathematical Sciences, Queen Mary,
University of London, London E1 4NS, U.K.
p.j.cameron@qmul.ac.uk

The group $\mathrm{PGL}(2,q)$ is the group of all linear fractional transformations $x\mapsto(ax+b)/(cx+d)$ of the projective line over $\mathrm{GF}(q)$, where $ad-bc\neq 0$.

The *covering radius* of a set S of permutations is the maximum, over all permutations $g\in S_n$, of $d(g,S)=\min_{h\in S}d(g,h)$, where d denotes Hamming distance.

The covering radius of $\mathrm{PGL}(2,q)$ is known in all cases where q is not congruent to 1 mod 6. (It is $q-2$ if q is even, and $q-3$ if q is odd.) For the excluded case, it is only known that the covering radius is between $q-5$ and $q-3$ inclusive.

Problem: Find the exact value.

Remark: This problem has a geometric formulation. What is the smallest s such that there is a set of $q+1$ points in the classical Minkowski plane (ruled quadric) of order q which meets every line in exactly one point and every conic in at most s points? The covering radius of $\mathrm{PGL}(2,q)$ is then $q+1-s$.

Problem 13 (BCC19.13) Disjoint intersections in Intersecting families.

Proposed by J. Talbot.

Correspondent: J. Talbot
Mathematical Institute, University of Oxford, 24–29 St Giles', Oxford OX1 3LB, U.K.
talbot@maths.ox.ac.uk

Let $X = \{1,2,\ldots,n\}$, and let $X^{(r)}$ be the set of all r-element subsets of X. If $\mathcal{A} \subseteq X^{(r)}$ is an intersecting family, define

$$\mathcal{A}\langle 1\rangle = \{x \in X : \exists A, B \in \mathcal{A} \text{ with } A \cap B = \{x\}\}.$$

Lovász [7] proved that, for $r \geq 1$, the number

$$\alpha_r = \max_{n \geq r}\{|\mathcal{A}\langle a\rangle| : \mathcal{A} \subset X^{(r)} \text{ is intersecting}\}$$

exists; with the best known bounds for α_r due to Tuza [9].

For $1 \leq k \leq r \leq n$, let

$$\mathcal{A}\langle k\rangle = \{C \in X^{(k)} : \exists A, B \in \mathcal{A} \text{ with } A \cap B = C\}.$$

Conjecture: If $A_1, A_2, \ldots, A_m \in \mathcal{A}\langle k\rangle$ are pairwise disjoint, then $m \leq \alpha_{r-k+1}$.

References

[1] B. Bollobas and I. Leader, Matchings and paths in the cube, *Discrete Applied Math.* **75** (1997), 1–8.

[2] O. V. Borodin and A. V. Kostochka, On an upper bound of a graph's chromatic number, depending on the graph's degree and density, *J. Combinatorial Theory* (B) **23** (1977), 247–250.

[3] G. Chartrand, H. Galvas, R. C. Vandell and F. Harary, The forced domination number of a graph, *J. Combinatorial Math. Combinatorial Computing* **25** (1997), 161–174.

[4] P. Erdős, On some extremal problems in graph theory, *Israel J. Math.* **3** (1965), 113–116.

[5] R. A. H. Gower, Critical sets in products of Latin squares, *Ars Combinatoria* **55** (2000), 293–317.

[6] W. H. Haemers and E. Spence, Enumeration of cospectral graphs, *Europ. J. Combinatorics*, to appear.

[7] L. Lovász, *Combinatorial Problems and Exercises*, North Holland, Amsterdam, New York, Oxford, 1979.

[8] N. J. A. Sloane (ed.), Online Encyclopaedia of Integer Sequences, `http://www.research.att.com/~njas/sequences/`

[9] Zs. Tuza, Critical hypergraphs and intersecting set-pair systems, *J. Combinatorial Theory* Ser. B, **39** (1985), 134–145.

[10] D. B. West (editor), Research Problems, *Discrete Math.* **257** (2002), 599–624.