Solutions to odd-numbered exercises

Peter J. Cameron, *Introduction to Algebra*, Chapter 5

5.1 All of the left and right module axioms are easily demonstrated by using the associative, distributive, zero, and additive inverse laws for matrices, while the extra bimodule axiom comes from the associative law. (Note that this bimodule is unital for both left and right actions.)

5.3 (a) Suppose that $M$ is generated as $R$-module by the element $m$. Define a map

$$\theta : R \rightarrow M$$

for all $r \in R$. We claim that $\theta$ is an $R$-module homomorphism. This involves checking that

$$m(r_1 + r_2) = mr_1 + mr_2,$$

$$m(r_1r_2) = (mr_1)r_2,$$

which are just module axioms (MM2) and (MM3). [In the case of the second axiom, we have to show that $(r_1r_2)\theta = (r_1\theta)r_2$. We are thinking of $r'$ as an element of the free $R$-module $R$, and $r_2$ as an element of the ring $R$ acting on this module.] The kernel of $\theta$ is $\{ r \in R : mr = 0 \}$. Clearly any element $r \in \text{Ann}(M)$ satisfies this condition. Conversely, since $m$ generates $M$, every element of $M$ has the form $mr'$ for some $r' \in R$; then $mr = 0$ implies $(mr')r = (mr)r' = 0$, so $r \in \text{Ann}(M)$. [Remember that our rings are now commutative with identity!] So the $\text{Ker}(\theta) = \text{Ann}(M)$. Also, since $m$ generates $M$, we have $\text{Im}(\theta) = M$. So $M \cong R/\text{Ann}(M)$ (as $R$-modules).

5.5 Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Then the characteristic polynomial of $A$ is $x^2 - (a + d)x + (ad - bc)$; and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = (a + d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

after some calculation.

The $3 \times 3$ case is left to you!

5.7 We know that $F^n$, as $F[x]$-module (as in Example 6 on page 185) is isomorphic to the direct sum of the modules $F(x)/(f_i(x))$, where $f_i(x)$ are the diagonal entries in the Smith normal form.

None of the polynomials $f_i(x)$ can be zero; for $F[x]/(0) = F[x]$ is an infinite-dimensional vector space over $F$, whereas $F^n$ is finite-dimensional. So $f_1(x), \ldots, f_n(x)$ are the invariant factors.

If $f_i(x) = 1$, then the corresponding summand is $F[x]/(1) = \{0\}$, and can be removed from the direct sum without changing it. So we are left with just the non-constant polynomials $f_i(x)$. 

1
5.9 (a) The set $X$ in question is just $\{ v \in V : vA = \lambda v \}$, since the eigenvectors are precisely the non-zero vectors in this set.

But, if $vA = \lambda v$ and $wA = \lambda w$, then

$$
(v + w)A = vA + wA = \lambda (v + w), \\
(cv)A = c(vA) = \lambda (cv);
$$

So $X$ passes the Subspace Test.

(b) Suppose that we have a linear dependence relation

$$c_1v_1 + \cdots + c_kv_k = 0,$$

where $c_1, \ldots, c_k$ are not all zero. We may assume that the number of non-zero elements among $c_1, \ldots, c_k$ is as small as possible for any linear dependence. The number of non-zero elements must be at least two. For suppose that all except $c_i$ were zero. Then we would have, say, $c_iv_i = 0$, contradicting the fact that $c_i \neq 0$ and $v_i \neq 0$ (as $v_i$ is an eigenvector).

Without loss of generality, we can suppose that $c_1$ and $c_2$ are non-zero. Multiplying the displayed equation by $A$, using the fact that $v_iA = \lambda_i v_i$, we obtain

$$\lambda_1c_1v_1 + \lambda_2c_2v_2 + \cdots + \lambda_kc_kv_k = 0.$$

Multiplying the displayed equation by $\lambda_1$, we obtain

$$\lambda_1c_1v_1 + \lambda_1c_2v_2 + \cdots + \lambda_1c_kv_k = 0.$$

Subtracting these two equations gives

$$(\lambda_2 - \lambda_1)c_2v_2 + \cdots + (\lambda_k - \lambda_1)c_kv_k = 0.$$

But this last relation is a linear dependence relation on $v_1, \ldots, v_k$ (since the coefficient of $v_2$ is $(\lambda_2 - \lambda_1)c_2 \neq 0$), with one fewer non-zero term than the equation with which we began. This contradicts the fact that we started with as few non-zero terms as possible.

5.11 (a) implies (b): If $\lambda$ is an eigenvalue of $A$, then there is a non-zero vector such that $vA = \lambda v$. Then $v(A - \lambda I) = O$, so $A - \lambda I$ is not invertible; so its determinant is zero. But this determinant is just $c(\lambda)$.

(b) implies (c): Suppose that $c(\lambda) = 0$, but $m(\lambda) \neq 0$. Then $\det(A - \lambda I) = 0$, so $A - \lambda I$ is not invertible; thus there exists a vector $v \neq 0$ with $vA = \lambda v$. But then an easy argument shows that $vm(A) = m(\lambda)v \neq 0$, contradicting the fact that $m(A) = 0$.

(c) implies (a): If $m(\lambda) = 0$, then $c(\lambda) = 0$ by the Cayley–Hamilton Theorem 5.21. Then just as in the case (b) implies (c), we see that there exists an eigenvector with eigenvalue $\lambda$.

Remark: In this proof, you see that we are covering some of the ground more than once. The direction of proof given in the book seems to be the most natural!
5.13 Let $d_1, \ldots, d_n$ be the diagonal elements in the Smith normal form of $A$. By Theorem 5.9,

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}.$$ 

We see immediately that, if some $d_i$ is zero, then $G$ is infinite, while if all $d_i$ are finite, then $|G|$ is just their product.

Now, when we apply an elementary row or column operation to a matrix, the effect on the determinant (if any) is to multiply it by $-1$ or a unit. In $\mathbb{Z}$, the only units are $+1$ and $-1$. So the determinant of the Smith normal form of $A$ (which is $d_1d_2\cdots d_n$) is equal, up to sign, to $\det(A)$, and we are done.