1 We have $G = S_3 \times S_3$, and $A = S_3 \times \{1\} = \{(g, 1) : g \in S_3\}$. One complement is $H_1 = \{1\} \times S_3 = \{(1, g) : g \in S_3\}$. It is clear that this is a complement. Moreover, $H_1$ commutes with $A$, so the action $\phi_1$ of $H_1$ on $A$ is trivial; the semidirect product $A \rtimes_{\phi_1} H_1$ is just the direct product $A \times H_1$.

Another complement is the diagonal subgroup $H_2 = \{(g, g) : g \in S_3\}$.

(We have $(g, h) \in A \cap H_2 \Rightarrow h = 1$ and $g = h$, so $A \cap H_2 = \{1\}$; and similarly $AH_2 = G$.) The action $\phi_2$ of $H_2$ on $A$ is the usual conjugation action of $S_3$ on itself, since

$$(g, g)^{-1}(x, 1)(g, g) = (g^{-1}xg, 1).$$

**Remark:** Since $\text{Aut}(S_3)$ is isomorphic to $S_3$, the semidirect product in the second case is the holomorph of $S_3$. So this holomorph is isomorphic to $S_3 \times S_3$.

2 (a) Suppose that $G$ is complete. Then $\text{Out}(G) = \{1\}$, so

$$\text{Aut}(G) = \text{Inn}(G) \cong G/Z(G) \cong G,$$

the last isomorphism holding because also $Z(G) = \{1\}$.

(b) As in the first question, let $A$ be the first direct factor in $G \times G$, let $H_1$ be the second direct factor, and let $H_2$ be the diagonal subgroup $\{(g, g) : g \in G\}$. Then each of $H_1$ and $H_2$ is a complement to $A$, so $G \times G \cong A \times H_1 \cong A \times H_2$; each of $A$, $H_1$ and $H_2$ is isomorphic to $G$, but the homomorphism $\phi_1 : H_1 \to \text{Aut}(A)$ is trivial (since the two direct factors commute) while the homomorphism $\phi_2 : H_2 \to \text{Aut}(A)$ is the identity map.

(c) It is trivial that $Z(S_n) = \{1\}$ for $n \geq 3$. Also we showed in lectures that $\text{Out}(S_n) = \{1\}$ for $n \geq 3, n \neq 6$. So for these values of $n$, $S_n$ is complete.
(d) An example of such a group is the dihedral group $D_8$. It is generated by two elements $g$ and $h$ satisfying $g^4 = 1$, $h^2 = 1$, $h^{-1}gh = g^{-1}$. It can be checked that the eight maps which send $g \mapsto g^a$ and $h \mapsto hg^b$ (for $a = \pm 1, b = 0, 1, 2, 3$) each extend to automorphisms of $G$; and these are the only possibilities, since $g$ must map to an element of order 4, and $h$ to an element of order 2 which is not a power of $g$. Moreover, if $s : g \mapsto g, h \mapsto hg$ and $t : g \mapsto g^{-1}, h \mapsto h$, then it can be checked that $s^4 = 1, t^2 = 1,$ and $t^{-1}st = s^{-1}$. So $s$ and $t$ generate a dihedral group of order 8.

But $D_8$ is not complete since $|Z(D_8)| = 2$.

3 A group $G$ is soluble if it has a chain of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{r-1} \triangleright G_r = \{1\}$$

with $G_i$ normal in $G$ and $G_{i-1}/G_i$ abelian for $i = 1, \ldots, r$.

(a) Since a cyclic group of prime order is abelian, a supersoluble group satisfies the condition to be soluble.

An example of a soluble group which is not supersoluble is $A_4$. Its only normal subgroups are $A_4, V_4$ and $\{1\}$, so the only possible series of normal subgroups consists of these three; and $V_4$ is abelian but not cyclic.

(b) Since a subgroup of the centre of a group is abelian and normal in the group, it is also clear that a nilpotent group satisfies the condition to be soluble.

An example of a soluble group which is not nilpotent is $S_3$. For the centre of a non-trivial nilpotent group is non-trivial, but $Z(S_3) = \{1\}$.

(c) Let $|G| = p^n$. We prove $G$ nilpotent by induction on $n$. The result is clear if $n = 0$. If $n > 0$, then $Z(G) \neq \{1\}$, and $G/Z(G)$ has order $p^m$ for some $m < n$. By the induction hypothesis, $G/Z(G)$ has a central series; the subgroups of $G$ corresponding to it (using the Correspondence Theorem) together with $Z(G)$ form a central series for $G$.

4 (a) $\text{AGL}(n, 2)$ is generated by the translations of $V = \mathbb{F}_2^n$ and the invertible linear maps. The translation group acts transitively, so it suffices to show that the stabiliser of the zero vector (which is $\text{GL}(n, 2)$) is doubly transitive.

Now, over $\mathbb{F}_2$, any two non-zero vectors are linearly independent (the only possible linear combination would be $v_1 + v_2 = 0$, implying that $v_1 = v_2$), so can be extended to a basis; and we can carry any basis to any other by an element of the general linear group. So $\text{GL}(n, 2)$ is doubly transitive, as required.

(b) $\text{AGL}(2, 2)$ has order $4|\text{GL}(2, 2)| = 4 \cdot 6 = 24$, and acts on the four vectors of $\mathbb{F}_2^2$, so is a subgroup of $S_4$. So $\text{AGL}(2, 2) \cong S_4$.

(c) $\text{AGL}(3, 2)$ acts on the eight points of $\mathbb{F}_2^3$, so is a subgroup of $S_8$. We need to show it is contained in $A_8$, that is, consists of even permutations. The translations are
products of four 2-cycles, so are even permutations. For the elements of \( \text{GL}(3, 2) \), either do this directly, or use the fact that if it were not so then \( A_8 \cap \text{GL}(3, 2) \) would be a subgroup of index 2 in \( \text{GL}(3, 2) \), hence normal, contradicting the simplicity of this group.

Now the index is \( \frac{1}{2} \frac{8!}{(8 \cdot 168)} = 15 \).

(d) The action of \( A_8 \) on the fifteen cosets of \( \text{AGL}(3, 2) \) is transitive. So we have to show that \( \text{AGL}(3, 2) \) is transitive on the other 14 cosets.

This group contains a Sylow 7-subgroup, which has order 7 and is generated by a product of two 7-cycles. (The only alternative would be that the generator is a single 7-cycle; then the Sylow 7-subgroup would lie in eight conjugates of \( \text{AGL}(3, 2) \), which is not possible.) So if the conclusion is false, then \( \text{AGL}(3, 2) \) itself would have two orbits of size 7. Pick one of these orbits and count the ordered pairs \((\alpha, \beta)\) where \( \beta \) lies in the specified orbit of the stabiliser of \( \alpha \); there are 105 such pairs. Since 105 is odd, these pairs cannot be interchanged by an element of \( A_8 \). But an element of order 2 in \( A_8 \) must interchange some pair of points, a contradiction.

An alternative argument would be that, if \( \text{AGL}(3, 2) \) acts on 7 points, the translation group must act trivially; and it cannot act trivially on the whole set since \( A_8 \) is simple. Thus not all orbits can have size 7 (or 1).