Queen Mary University of London

MTHM024/MTH714U Group Theory

Problem Sheet 7 Solutions

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1 We have $G = S_3 \times S_3$, and $A = S_3 \times \{1\} = \{(g, 1) : g \in S_3\}$.

One complement is $H_1 = \{1\} \times S_3 = \{(1, g) : g \in S_3\}$. It is clear that this is a complement. Moreover, H_1 commutes with A, so the action ϕ_1 of H_1 on A is trivial; the semidirect product $A \rtimes_\phi H_1$ is just the direct product $A \times H_1$.

Another complement is the *diagonal* subgroup

$$H_2 = \{(g, g) : g \in S_3\}.$$

(We have $(g,h) \in A \cap H_2 \Rightarrow h = 1$ and $g = h$, so $A \cap H_2 = \{1\}$; and similarly $AH_2 = G$.) The action ϕ_2 of H_2 on A is the usual conjugation action of S_3 on itself, since

$$(g, g)^{-1}(x, 1)(g, g) = (g^{-1}xg, 1).$$

Remark: Since $\mathrm{Aut}(S_3)$ is isomorphic to S_3, the semidirect product in the second case is the holomorph of S_3. So this holomorph is isomorphic to $S_3 \times S_3$.

2 (a) Suppose that G is complete. Then $\mathrm{Out}(G) = \{1\}$, so

$$\mathrm{Aut}(G) = \mathrm{Inn}(G) \cong G/Z(G) \cong G,$$

the last isomorphism holding because also $Z(G) = \{1\}$.

(b) As in the first question, let A be the first direct factor in $G \times G$, let H_1 be the second direct factor, and let H_2 be the diagonal subgroup $\{(g, g) : g \in G\}$. Then each of H_1 and H_2 is a complement to A, so $G \times G \cong A \rtimes H_1 \cong A \rtimes H_2$; each of A, H_1 and H_2 is isomorphic to G, but the homomorphism $\phi_1 : H_1 \to \mathrm{Aut}(A)$ is trivial (since the two direct factors commute) while the homomorphism $\phi_2 : H_2 \to \mathrm{Aut}(A)$ is the identity map.

(c) It is trivial that $Z(S_n) = \{1\}$ for $n \geq 3$. Also we showed in lectures that $\mathrm{Out}(S_n) = \{1\}$ for $n \geq 3$, $n \neq 6$. So for these values of n, S_n is complete.

(d) An example of such a group is the dihedral group D_8. It is generated by two elements g and h satisfying $g^4 = 1$, $h^2 = 1$, $h^{-1}gh = g^{-1}$. It can be checked that the eight maps which send $g \mapsto g^a$ and $h \mapsto hg^b$ (for $a = \pm 1$, $b = 0,1,2,3$) each extend to automorphisms of G; and these are the only possibilities, since g must map to an element of order 4, and h to an element of order 2 which is not a power of g. Moreover, if $s : g \mapsto g, h \mapsto hg$ and $t : g \mapsto g^{-1}, h \mapsto h$, then it can be checked that $s^4 = 1$, $t^2 = 1$, and $t^{-1}st = s^{-1}$. So s and t generate a dihedral group of order 8.

But D_8 is not complete since $|Z(D_8)| = 2$.

3 A group G is soluble if it has a chain of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{r-1} \triangleright G_r = \{1\}$$

with G_i normal in G and G_{i-1}/G_i abelian for $i = 1, \ldots, r$.

(a) Since a cyclic group of prime order is abelian, a supersoluble group satisfies the condition to be soluble.

An example of a soluble group which is not supersoluble is A_4. Its only normal subgroups are A_4, V_4 and $\{1\}$, so the only possible series of normal subgroups consists of these three; and V_4 is abelian but not cyclic.

(b) Since a subgroup of the centre of a group is abelian and normal in the group, it is also clear that a nilpotent group satisfies the condition to be soluble.

An example of a soluble group which is not nilpotent is S_3. For the centre of a non-trivial nilpotent group is non-trivial, but $Z(S_3) = \{1\}$.

(c) Let $|G| = p^n$. We prove G nilpotent by induction on n. The result is clear if $n = 0$. If $n > 0$, then $Z(G) \neq \{1\}$, and $G/Z(G)$ has order p^m for some $m < n$. By the induction hypothesis, $G/Z(G)$ has a central series; the subgroups of G corresponding to it (using the Correspondence Theorem) together with $Z(G)$ form a central series for G.

4 (a) $\mathrm{AGL}(n,2)$ is generated by the translations of $V = \mathbb{F}_2^n$ and the invertible linear maps. The translation group acts transitively, so it suffices to show that the stabiliser of the zero vector (which is $\mathrm{GL}(n,2)$) is doubly transitive.

Now, over $\mathbb{F}_2$, any two non-zero vectors are linearly independent (the only possible linear combination would be $v_1 + v_2 = 0$, implying that $v_1 = v_2$), so can be extended to a basis; and we can carry any basis to any other by an element of the general linear group. So $\mathrm{GL}(n,2)$ is doubly transitive, as required.

(b) $\mathrm{AGL}(2,2)$ has order $4|\mathrm{GL}(2,2)| = 4 \cdot 6 = 24$, and acts on the four vectors of $\mathbb{F}_2^2$, so is a subgroup of S_4. So $\mathrm{AGL}(2,2) \cong S_4$.

(c) $\mathrm{AGL}(3,2)$ acts on the eight points of $\mathbb{F}_2^3$, so is a subgroup of S_8. We need to show it is contained in A_8, that is, consists of even permutations. The translations are products of four 2-cycles, so are even permutations. For the elements of $\mathrm{GL}(3,2)$, either do this directly, or use the fact that if it were not so then $A_8 \cap \mathrm{GL}(3,2)$ would be a subgroup of index 2 in $\mathrm{GL}(3,2)$, hence normal, contradicting the simplicity of this group.

Now the index is $\frac{1}{2}\,8!/(8\cdot 168) = 15$.

(d) The action of A_8 on the fifteen cosets of $\mathrm{AGL}(3,2)$ is transitive. So we have to show that $\mathrm{AGL}(3,2)$ is transitive on the other 14 cosets.

This group contains a Sylow 7-subgroup, which has order 7 and is generated by a product of two 7-cycles. (The only alternative would be that the generator is a single 7-cycle; then the Sylow 7-subgroup would lie in eight conjugates of $\mathrm{AGL}(3,2)$, which is not possible.) So if the conclusion is false, then $\mathrm{AGL}(3,2)$ itself would have two orbits of size 7. Pick one of these orbits and count the ordered pairs (α,β) where β lies in the specified orbit of the stabiliser of α; there are 105 such pairs. Since 105 is odd, these pairs cannot be interchanged by an element of A_8. But an element of order 2 in A_8 must interchange some pair of points, a contradiction.

An alternative argument would be that, if $\mathrm{AGL}(3,2)$ acts on 7 points, the translation group must act trivially; and it cannot act trivially on the whole set since A_8 is simple. Thus not all orbits can have size 7 (or 1).