Queen Mary University of London

MTHM024/MTH714U — Group Theory

Solutions 2 — **November 2011**

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1 (a) Any permutation can be written as a product of disjoint cycles, so it is enough to show that any cycle can be written as a product of transpositions. Check directly that

$$(1,2,3,\ldots,n) = (1,2)(1,3)\cdots(1,n).$$

(b) The relation is clearly reflexive and symmetric. To prove transitivity, suppose that $i \sim j$ and $j \sim k$. The cases where two of i,j,k are equal are straightforward, so suppose that they are all distinct. Then $(i,j),(j,k) \in G$, and so $(i,j)(j,k)(i,j) = (i,k) \in G$, whence $i \sim k$.

By definition, if Δ is an equivalence class, then G contains all transpositions (i,j) for $i,j \in \Delta$, $i \neq j$; these generate the symmetric group on Δ, fixing every point outside Δ.

(c) Let N be the subgroup of G generated by its transpositions. Since the set of transpositions is closed under conjugation, N is a normal subgroup. Now N fixes each equivalence class (as a set), and contains all the permutations fixing each class. Any such permutation is uniquely expressible as a product of permutations on the equivalence classes; so N is a direct product as claimed.

2 (a) We are looking for Sylow p-subgroups for $p = 5,3,2$; they should have orders $5,3,8$ respectively. It is enough to give an example of each.

$p = 5$: the cyclic group generated by a 5-cycle;

$p = 3$: the cyclic group generated by a 3-cycle;

$p = 2$: the dihedral group of symmetries of a square, fixing the remaining point.

There are 24 elements of order 5, hence $24/4 = 6$ Sylow 5-subgroups; 20 elements of ordedr 3, hence $20/2 = 10$ Sylow 3-subgroups. To count the Sylow 2-subgroups we note that, by the conjugacy part of Sylow's theorem, they are all symmetry groups of squares, so we have to count the number of ways of labelling a square and an isolated point. There are 5 choices for the isolated point, and 3 ways of labelling the square; so 15 Sylow 2-subgroups.

(b) The conjugacy classes in S_5 have sizes 1 (the identity), 10 (the transpositions), 15 (products of two transpositions), 20 (the 3-cycles), 20 (the products of a 2-cycle and a 3-cycle), 30 (the 4-cycles), and 24 (the 5-cycles). How can we choose some of these, including the identity, to have size dividing 120? There are trivial solutions corresponding to the identity and the whole group; what others are there? A little thought shows that the numerical solutions are $1 + 24 + 15$ or $1 + 24 + 15 + 20$. A subgroup containing elements which are the product of a 2-cycle and a 3-cycle would also contain their squares, which are 3-cycles; so the 20 must be the class of 3-cycles. Now it is easy to see that we can write a 3-cycle as the product of two double transpositions; so if we include 15 we must also include 20. So the only possibility is to use all the even permutations, obtaining A_5.

Thus the only normal subgroups are $\{1\}$, A_5 and S_5.

3 (a) The number of Sylow 5-subgroups of a group of order 40 is congruent to 1 (mod 5) and divides 8, so is 1; thus a Sylow 5-subgroup is normal.

(b) The number of Sylow 7-subgroups of a group of order 84 is congruent to 1 (mod 7) and divides 12, so is 1; thus a Sylow 7-subgroup is normal.