Queen Mary
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MTHM024/MTH714U Group Theory

Solutions 1 October 2011

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1 (a) By the Subgroup Test, we have to show that $H \cap K$ is non-empty (which it is, as it contains the identity), and that if $x, y \in H \cap K$, then $xy^{-1} \in H \cap K$. This holds because $x, y \in H$, so $xy^{-1} \in H$ (as H is a subgroup), and similarly $xy^{-1} \in K$.

(b) We claim that, if $x \in HK$, then x can be written as hk (with $h \in H$ and $k \in K$) in exactly $|H \cap K|$ ways. Given one such expression $x = hk$, we have $x = (hg^{-1})(gk)$ for all $g \in H \cap K$, giving $|H \cap K|$ expressions, Conversely, if $x = h'k'$ is any such expression, then $hk = h'k'$, so $(h')^{-1}h = k'k_{-1} = g$, say, so $h' = hg^{-1}$ and $k' = gk$. So we have found all such expressions.

Hence $|HK| = |H| \cdot |K| / |H \cap K|$, since by counting the pairs (h, k) we overcount by a factor of $|H \cap K|$.

(c) Clearly HK is non-empty. If $h_1k_1, h_2k_2 \in HK$, then

$$\begin{aligned} (h_1k_1)(h_2k^2)^{-1} &= h_1kh_2^{-1} && \text{where } k = k_1k_2^{-1} \\ &= h_1h_3k && \text{where } h_3 = kh_2^{-1}k^{-1} \in kHk^{-1} = k \\ &\in HK, \end{aligned}$$

so HK is a subgroup.

(d) Let $G = S_3$, and let H and K be the subgroups of order 2 generated by $(1, 2)$ and $(1, 3)$ respectively. Then $|HK| = 4$, and so HK cannot be a subgroup of a group of order 6, by Lagrange's Theorem.

2 We have

$$\begin{aligned} (q_1 + q_2)\theta &= \mathrm{e}^{2\pi \mathrm{i}(q_1+q_2)} \\ &= \mathrm{e}^{2\pi \mathrm{i} q_1} \cdot \mathrm{e}^{2\pi \mathrm{i} q_2} \\ &= q_1\theta \cdot q_2\theta, \end{aligned}$$

so θ is a homomorphism.

Since every root of unity has the form $\mathrm{e}^{2\pi \mathrm{i} q}$ for some rational number q, θ is onto. Its kernel is

$$\{q \in \mathbb{Q} : \mathrm{e}^{2\pi \mathrm{i} q} = 1\} = \mathbb{Z}.$$

So $\mathbb{Q}/\mathbb{Z} \cong A$, by the First Isomorphism Theorem.

Every element of A has finite order (the order of $\mathrm{e}^{2\pi \mathrm{i} q}$ is the denominator of q), while every non-zero element of the infinite cyclic group has infinite order. So they are not isomorphic.

3 This question was discussed in the class; here is a summary.

Tetrahedron: every symmetry induces a permutation on the four vertices; if we number them $\{1,2,3,4\}$, then we have homomorphisms from the rotation and symmetry groups to S_4, which are obviously one-to-one. Comparing orders we see that they symmetry group is S_4; the rotation group has order 12, and the only subgroup of S_4 of order 12 is A_4. (Alternatively, list directly the rotations as permutations and observe that they are all even permutations.)

Cube: We construct an action of the symmetry group on the set of four diagonals of the cube (each joining a vertex to its opposite). This gives a homomorphism from the symmetry group to S_4; its kernel is easily seen to be $\{\pm I\}$. So restricted to the rotation group, the action is faithful, and the rotation group is S_4.

In the symmetry group G, we now have two normal subgroups: the rotation group $H \cong S_4$, of order 24; and the group $K = \{\pm I\}$, of order 2. Their intersection is the identity and their product is G. So $G \cong H \times K$.

4 (a) Split the n points up into a_1 sets of size p^i for $i = 0, \ldots, r$, and let G be the group fixing each of these sets. It is easily seen that G is the direct product of symmetric groups of the appropriate sizes.

Some slightly tedious number theory (which we will need later) shows that the power of p dividing $n!$ is the same as the power of p dividing $|G|$.

(b) The power of p dividing $p^i!$ is as claimed: if we write out the factorial as a product, there are p^{i-1} terms which are multiples of p, of which p^{i-2} are multiples of p^2, and so on.

Given a set of size p^i, choose partitions $\pi_1, \pi_2, \ldots, \pi_{i-1}$ where pi_j has p^j parts of size p^{i-j}, and each partition refines the one before. Now consider permutations which fix all these partitions, and permute the parts of π_{j+1} in each part of π_j cyclically. This has the required order.

(c) Take the direct product of Sylow p-subgroups of S_{p^i} for each i to get a Sylow p-subgroup of G. By our remark in the first part, this is also a Sylow p-subgroup of S_n.

(d) Given any finite group H of order n, by Cayley's Theorem we can embed H into the symmetric group S_n, which has a Sylow p-subgroup. By Sylow's Lemma, H has a Sylow p-subgroup.