MTHM024/MTH714U **Group Theory**

Supplementary Problem Sheet **Autumn 2011**

This sheet contains miscellaneous problems on the first half of the course.

1 Consider the group of order 168 on the Introductory Problem Sheet. We already saw that this group acts doubly transitively on the seven points of the Fano plane. We also saw that this group has eight Sylow 7-subgroups. Prove that it acts doubly transitively on the set of Sylow 7-subgroups.

2 Show that there is no simple group of order 120.

3 Suppose that a and b are elements of a finite group G satisfying $a^2 = b^2 = 1$.

(a) Show that $\langle a, b\rangle$ is a dihedral group D_{2m} for some m.

(b) If m is odd, show that a and b are conjugate in G.

(c) If m is even, show that G contains an element c satisfying $c^2 = 1$ which commutes with both a and b.

4 Let G be a group of order $2m$, where m is odd and $m > 1$. Prove that G is not simple. [Hint: Consider the action of G on itself by right multiplication; show that this action contains an odd permutation.]

5 Show that the outer automorphism of S_6 interchanges the conjugacy classes of types $[1,1,1,1,2]$ and $[2,2,2]$, those of types $[1,1,1,3]$ and $[3,3]$, and those of types $[1,2,3]$ and $[6]$, and fixes the other classes.

6 Don't tackle parts (b) and (c) of this question unless you have met primitive roots (e.g. in a number theory course). Let $U(n)$ be the group of units of the ring Z_n of integers mod n.

(a) Prove that, if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then

$$U(n) \cong U(p_1^{a_1}) \times U(p_2^{a-2}) \times \cdots \times U(p_r^{a_r}).$$

[Hint: Chinese Remainder Theorem.]

(b) Prove that, if p is prime, then $U(p) \cong C_{p-1}$.

(c) Prove that, if p is prime and $a \geq 1$, then $U(p^a) \cong C_{(p-1)p^{a-1}}$.

(d) Prove that $U(2^a) \cong C_2 \times C_{2^{a-2}}$ for $a \geq 2$. [Hint: the factors are generated by the units -1 and 5.]