Queen Mary
University of London

M. Sc. Examination by course unit 2011

MTHM024 Group Theory

Duration: 3 hours

Date and time: 13 May 2011, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): P. J. Cameron

Question 1 State and prove the *Jordan–Hölder theorem* on composition factors of a finite group. [17]

The following are two properties which may or may not hold in a group of order 120:

A: G has a normal subgroup N with $N \cong A_5$, $G/N \cong C_2$;

B: G has a normal subgroup N with $N \cong C_2$, $G/N \cong A_5$.

Give examples of groups satisfying

(a) A and B; (b) A but not B; (c) B but not A; (d) neither A nor B.

You should indicate why your groups have the required properties, but detailed proofs are not required. [8]

Question 2 Define the groups $\mathrm{GL}(n,F)$, $\mathrm{SL}(n,F)$, $\mathrm{PGL}(n,F)$, and $\mathrm{PSL}(n,F)$, where $n \geq 2$ and F is a field. [4]

State without proof the orders of these groups when F is a finite field with q elements. [4]

State without proof which of the groups $\mathrm{PSL}(n,F)$ are simple. [3]

Prove that $\mathrm{PSL}(2,3) \cong A_4$ and $\mathrm{PSL}(2,5) \cong A_5$, where A_n is the alternating group of degree n. [14]

Question 3 Define a *Sylow p-subgroup* of the finite group G, where p is prime. [3]

State Sylow's theorems. [5]

Let $G = \mathrm{GL}(3,p)$ be the group of invertible 3×3 matrices over $\mathbb{Z}_p$, the field of integers mod p, where p is prime. Let

$$P = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{Z}_p \right\}, \qquad Q = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} : a,b,c \in \mathbb{Z}_p \right\}.$$

Show that P and Q are Sylow p-subgroups of G, and find an element $g \in G$ such that $g^{-1}Pg = Q$. [11]

Are P and Q isomorphic? Are they abelian? [6]

Question 4 (a) What is an *action* of the group G on the set Ω? If G acts on Ω, what are its *orbits*? [4]

(b) Give a formula for the number of orbits in the case when G and Ω are finite. [3]

(c) What is a *transitive* action? What is a *primitive* action? [4]

(d) If G acts primitively on Ω and $N \trianglelefteq G$, show that the action of N is either trivial or transitive. [6]

(e) Show that the action of S_4 on the set of 2-element subsets of $\{1,2,3,4\}$ is imprimitive. Hence find a homomorphism from S_4 onto S_3. [8]

Question 5 (a) What is a *soluble* group? [3]

(b) Prove that all dihedral groups are soluble. [4]

(c) Prove that every group of order 48 is soluble. (You may assume that groups of order smaller than 48 are soluble, but any theorems you use about soluble groups should be stated clearly.) [10]

(d) Show that a direct product of soluble groups is soluble. [8]

Question 6 Let H be a subgroup of the group G. Define the *normaliser* $N_G(H)$ of H in G. [2]

If G is the dihedral group of order 8 (the group of symmetries of a square), t is a reflection in G, and $H = \langle t \rangle$, find the order of $N_G(H)$. [5]

Prove that

(a) if G has prime power order and $H < G$, then $H < N_G(H)$; [9]

(b) if P is a Sylow p-subgroup of G and $N_G(P) \leq H$, then $N_G(H) = H$. [9]

Question 7 Let G be a group with an abelian normal subgroup A having a complement H.

(a) Prove that $G/A \cong H$. [5]

(b) Show how the action of H on A by conjugation gives rise to a homomorphism $\phi : H \to \mathrm{Aut}(A)$. [5]

(c) Show that G can be reconstructed from knowledge of A, H and ϕ. [5]

(d) Show that, if the image of ϕ is $\{1\}$, then $G \cong A \times H$. [4]

(e) What further information is needed to reconstruct G if the normal subgroup A is abelian but does not necessarily have a complement? (Details not required.) [6]

End of Paper