Queen Mary
University of London

MTHM024/MTH714U **Group Theory**

Problem Sheet 7 **1 December 2011**

The problem sheets in this course are for "formative assessment" only; there is no coursework component in the assessment.

Any work handed in by the lecture on the date at the top of the sheet will be marked and returned to you in the next week's lecture.

1 Let G be the group $S_3 \times S_3$. Let A denote the first direct factor. Find two complements to A in G, one of which is normal and the other is not. Hence show that this group can be expressed as $S_3 \rtimes_\phi S_3$ with two different homomorphisms ϕ from S_3 to $\mathrm{Aut}(S_3)$. (Note that $\mathrm{Aut}(S_3)$ is isomorphic to S_3.)

2 A group G is said to be *complete* if $Z(G) = \{1\}$ and $\mathrm{Out}(G) = \{1\}$.

(a) Show that, if G is complete, then $\mathrm{Aut}(G) \cong G$.

(b) Following the argument of the preceding question, show that, if G is complete, then $G \times G$ can be expressed as $G \rtimes_\phi G$ for two very different homomorphisms $\phi : G \to \mathrm{Aut}(G)$.

(c) Give an example of a complete group G not equal to S_3.

(d) Give an example of a group G satisfying $\mathrm{Aut}(G) \cong G$ for which G is not complete.

3 (a) A finite group G is said to be *supersoluble* if it has a chain

$$G = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{r-1} \triangleleft G_r = \{1\}$$

such that, for $i = 1, \ldots, r$, the subgroup G_i is normal in G (not just in G_{i-1}) and G_{i-1}/G_i is cyclic of prime order. Prove that a supersoluble group is soluble, and give an example of a soluble group which is not supersoluble.

(b) A finite group G is said to be *nilpotent* if it has a chain

$$G = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{r-1} \triangleleft G_r = \{1\}$$

such that, for $i = 1, \ldots, r$, the subgroup G_i is normal in G and G_{i-1}/G_i is contained in the centre of G/G_i. Prove that a nilpotent group is soluble, and give an example of a soluble group which is not nilpotent.

(c) Show that a group whose order is a prime power is nilpotent.

4 Recall that the *affine group* $\mathrm{AGL}(n, p)$ is the semidirect product of $(C_p)^n$ by its automorphism group $\mathrm{GL}(n, p)$. We regard $(C_p)^n$ as the additive group of the n-dimensional vector space over the field $\mathbb{F}_p$.

(a) Show that the affine group $\mathrm{AGL}(n, 2)$ is a triply transitive permutation group of degree 2^n. [Hint: The stabiliser of the zero vector is $\mathrm{GL}(n, 2)$; show that this group is doubly transitive on non-zero vectors.]

(b) Show that $\mathrm{AGL}(2, 2)$ is isomorphic to the symmetric group S_4.

(c) Show that the affine group $\mathrm{AGL}(3, 2)$ is contained in the alternating group A_8 as a subgroup of index 15.

(d) Show that A_8 acts doubly transitively on the 15 elements of $\cos(\mathrm{AGL}(3, 2), A_8)$.

Remark: In fact, A_8 is isomorphic to $\mathrm{GL}(4, 2)$, and this action on 15 points is isomorphic to the action on the non-zero vectors of the 4-dimensional vector space.