1 Let $G$ be the group $S_3 \times S_3$. Let $A$ denote the first direct factor. Find two comple-
ments to $A$ in $G$, one of which is normal and the other is not. Hence show that this
group can be expressed as $S_3 \rtimes \phi S_3$ with two different homomorphisms $\phi$ from $S_3$ to
Aut($S_3$). (Note that Aut($S_3$) is isomorphic to $S_3$.)

2 A group $G$ is said to be complete if $Z(G) = \{1\}$ and Out($G$) = $\{1\}$.

(a) Show that, if $G$ is complete, then Aut($G$) $\cong G$.

(b) Following the argument of the preceding question, show that, if $G$ is complete,
then $G \times G$ can be expressed as $G \rtimes \phi G$ for two very different homomorphisms
$\phi : G \rightarrow \text{Aut}(G)$.

(c) Give an example of a complete group $G$ not equal to $S_3$.

(d) Give an example of a group $G$ satisfying Aut($G$) $\cong G$ for which $G$ is not com-
plete.

3 (a) A finite group $G$ is said to be supersoluble if it has a chain

$$G = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{r-1} \triangleleft G_r = \{1\}$$

such that, for $i = 1, \ldots, r$, the subgroup $G_i$ is normal in $G$ (not just in $G_{i-1}$) and
$G_{i-1}/G_i$ is cyclic of prime order. Prove that a supersoluble group is soluble, and
give an example of a soluble group which is not supersoluble.
(b) A finite group $G$ is said to be \textit{nilpotent} if it has a chain

$$G = G_0 \lhd G_1 \lhd \cdots \lhd G_{r-1} \lhd G_r = \{1\}$$

such that, for $i = 1, \ldots, r$, the subgroup $G_i$ is normal in $G$ and $G_{i-1}/G_i$ is contained in the centre of $G/G_i$. Prove that a nilpotent group is soluble, and give an example of a soluble group which is not nilpotent.

(c) Show that a group whose order is a prime power is nilpotent.

4 Recall that the \textit{affine group} $AGL(n, p)$ is the semidirect product of $(C_p)^n$ by its automorphism group $GL(n, p)$. We regard $(C_p)^n$ as the additive group of the $n$-dimensional vector space over the field $\mathbb{F}_p$.

(a) Show that the affine group $AGL(n, 2)$ is a triply transitive permutation group of degree $2^n$. [Hint: The stabiliser of the zero vector is $GL(n, 2)$; show that this group is doubly transitive on non-zero vectors.]

(b) Show that $AGL(2, 2)$ is isomorphic to the symmetric group $S_4$.

(c) Show that the affine group $AGL(3, 2)$ is contained in the alternating group $A_8$ as a subgroup of index 15.

(d) Show that $A_8$ acts doubly transitively on the 15 elements of $\cos(AGL(3, 2), A_8)$.

\textbf{Remark:} In fact, $A_8$ is isomorphic to $GL(4, 2)$, and this action on 15 points is isomorphic to the action on the non-zero vectors of the 4-dimensional vector space.