MTHM024/MTH714U **Group Theory**

Problem Sheet 6 **24 November 2011**

The problem sheets in this course are for "formative assessment" only; there is no coursework component in the assessment.

Any work handed in by the lecture on the date at the top of the sheet will be marked and returned to you in the next week's lecture.

1 Consider $G = \mathrm{PGL}(2,F)$ as the group of linear fractional transformations $x \mapsto (ax+b)/(cx+d)$ of $F \cup \{\infty\}$ with $ad-bc \neq 0$.

(a) Show that G acts transitively.

(b) Show that the stabiliser of ∞ is the "affine group" of all transformations $x \mapsto ax+b$ with $a \neq 0$. Deduce that G is doubly transitive.

(c) Show that the stabiliser of ∞ and 0 is the multiplicative group of F. Deduce that G is triply transitive and that the stabiliser of any three points is the identity.

2 Show that there is no simple group whose order is the product of three distinct primes.

3 Let V consist of the $\binom{6}{2} = 15$ 2-element subsets of $\{1,2,3,4,5,6\}$, together with one extra symbol 0. Define an operation $\oplus$ on V by the rules

- $v \oplus 0 = 0 \oplus v = v$ and $v \oplus v = 0$ for any $\in V$;
- $\{a,b\} \oplus \{a,c\} = \{b,c\}$ for all distinct $a,b,c \in \{1,\ldots,6\}$;
- $\{a,b\} \oplus \{c,d\} = \{e,f\}$ if $\{a,\ldots,f\} = \{1,\ldots,6\}$.

Prove that $(V,\oplus)$ is an elementary abelian 2-group of order 16, that is, the additive group of a 4-dimensional vector space over $\mathbb{F}_2$. Deduce that S_6 is a subgroup of $\mathrm{GL}(4,2)$. What is its index?

The next two questions are more difficult!

4 This question outlines a proof that any simple group of order 168 is isomorphic to $\mathrm{PSL}(2,7)$. Let G be a simple group of order 168.

(a) Show that G has 8 Sylow 7-subgroups, and that the normaliser of one such subgroup, say P, has order 21.

(b) Hence show that G acts doubly transitively on a set of 8 points, and the stabiliser of a point acts as the group

$$N = \{x \mapsto ax + b : a \in \{1,2,4\}, b \in \mathbb{Z}_7\}$$

of $\mathbb{Z}_7$. Deduce that the identity and the two maps $x \mapsto 2x$ and $x \mapsto 4x$ form a Sylow 3-subgroup Q of G.

(c) Let the stabilised point be named ∞. Show that there is an element t of order 2 in G which interchanges ∞ and normalises Q.

(d) Show that t is an even permutation, and deduce that it must interchange the two sets $\{1,2,4\}$ and $\{3,5,6\}$.

(e) By laborious computation (which you may omit), show that necessarily $t = (\infty,0)(1,6)(2,3)(4,5)$; in other words, t is the map $x \mapsto -1/x$.

(f) Show that N and t generate G.

(g) Now every element of G lies in $\mathrm{PSL}(2,7)$ (the group of linear fractional transformations of $\{\infty\} \cup \mathbb{Z}_7$. By comparing orders, $G = \mathrm{PSL}(2,7)$.

5 Let A, B and C be finite abelian groups. Show that the following are equivalent:

(a) A has a s subgroup isomorphic to B with quotient isomorphic to C;

(b) A has a s subgroup isomorphic to C with quotient isomorphic to B;

Show that this equivalence is false for

- infinite abelian groups;
- non-abelian groups.