Queen Mary
University of London

MTHM024/MTH714U Group Theory

Problem Sheet 5 17 November 2010

The problem sheets in this course are for "formative assessment" only; there is no coursework component in the assessment.

Any work handed in by the lecture on the date at the top of the sheet will be marked and returned to you in the next week's lecture.

1 (a) Construct addition and multiplication tables for a field with eight elements. [I hope you have met this before, and that this question is revision.]

(b) Prove that any two fields with eight elements are isomorphic. [*Hint:* you probably used an irreducible polynomial of degree 3 over $\mathbb{Z}_2$ in your construction: there are two such polynomials. If you use one polynomial in the construction, show that the field you construct also contains a root of the other polynomial.]

2 Let G be a subgroup of G. Let $N_G(H)$ be the *normaliser* of H in G, the largest subgroup of G in which H is contained as a normal subgroup. Alternatively,

$$N_G(H) = \{g \in G : g^{-1}Hg = H\}.$$

(a) Prove that, in the action of G on the coset space $\cos(H,G)$, a coset Hg is fixed by H if and only if $g \in N_G(H)$.

(b) Suppose that $|G| = p^n$, where p is prime, and that $H < G$. Prove that $H < N_G(H)$. (Recall that $H < G$ means "H is a subgroup of G and $H \neq G$".)

3 Show that the only element in the group $\mathrm{SL}(2,F)$, where F is a field whose characteristic is not 2, is $-I$.

Deduce that

(a) $\mathrm{SL}(2,F)$ does not contain a subgroup isomorphic to $\mathrm{PSL}(2,F)$;

(b) if $F = \mathrm{GF}(q)$ with $q > 3$, then the only composition series for $\mathrm{SL}(2,q)$ is $\{I\} \lhd \{\pm I\} \lhd \mathrm{SL}(2,q)$;

(c) $\mathrm{SL}(2,q)$ is not isomorphic to $C_2 \times \mathrm{PSL}(2,q)$.

4 This question is quite challenging!

Let G be a group whose order is a power of the prime p. Let N be the subgroup of G generated by pth powers of all elements and commutators of all pairs of elements.

(a) Prove that N is a normal subgroup of G, and that G/N is an elementary abelian p-group.

(b) Prove that, if K is any normal subgroup such that G/K is an elementary abelian p-group, then $N \leq K$.

(c) Prove that every maximal subgroup of G has index p and is normal.

(d) Prove that N is the intersection of all maximal subgroups of G.

(e) Prove that, if the cosets $Ng_1, \ldots, Ng_r$ generate G/N, then the elements $g_1, \ldots, g_r$ generate G.