1. (a) Construct addition and multiplication tables for a field with eight elements. [I hope you have met this before, and that this question is revision.]

(b) Prove that any two fields with eight elements are isomorphic. [Hint: you probably used an irreducible polynomial of degree 3 over \( \mathbb{Z}_2 \) in your construction: there are two such polynomials. If you use one polynomial in the construction, show that the field you construct also contains a root of the other polynomial.]

2. Let \( G \) be a subgroup of \( G \). Let \( N_G(H) \) be the normaliser of \( H \) in \( G \), the largest subgroup of \( G \) in which \( H \) is contained as a normal subgroup. Alternatively, \( N_G(H) = \{ g \in G : g^{-1}Hg = H \} \).

   (a) Prove that, in the action of \( G \) on the coset space \( \cos(H, G) \), a coset \( Hg \) is fixed by \( H \) if and only if \( g \in N_G(H) \).

   (b) Suppose that \( |G| = p^n \), where \( p \) is prime, and that \( H < G \). Prove that \( H < N_G(H) \). (Recall that \( H < G \) means “\( H \) is a subgroup of \( G \) and \( H \neq G \).”)

3. Show that the only element in the group \( \text{SL}(2,F) \), where \( F \) is a field whose characteristic is not 2, is \(-I\).

   Deduce that

   (a) \( \text{SL}(2,F) \) does not contain a subgroup isomorphic to \( \text{PSL}(2,F) \);

   (b) if \( F = \text{GF}(q) \) with \( q > 3 \), then the only composition series for \( \text{SL}(2,q) \) is \( \{I\} \triangleleft \{ \pm I \} \triangleleft \text{SL}(2,q) \);

   (c) \( \text{SL}(2,q) \) is not isomorphic to \( C_2 \times \text{PSL}(2,q) \).
This question is quite challenging!

Let $G$ be a group whose order is a power of the prime $p$. Let $N$ be the subgroup of $G$ generated by $p$th powers of all elements and commutators of all pairs of elements.

(a) Prove that $N$ is a normal subgroup of $G$, and that $G/N$ is an elementary abelian $p$-group.

(b) Prove that, if $K$ is any normal subgroup such that $G/K$ is an elementary abelian $p$-group, then $N \leq K$.

(c) Prove that every maximal subgroup of $G$ has index $p$ and is normal.

(d) Prove that $N$ is the intersection of all maximal subgroups of $G$.

(e) Prove that, if the cosets $Ng_1, \ldots, Ng_r$ generate $G/N$, then the elements $g_1, \ldots, g_r$ generate $G$. 