Queen Mary University of London

MTHM024/MTH714U Group Theory

Problem Sheet 4 — 10 November 2011

The problem sheets in this course are for "formative assessment" only; there is no coursework component in the assessment.

Any work handed in by the lecture on the date at the top of the sheet will be marked and returned to you in the next week's lecture.

1 Let $n \geq 2$. Let G be the symmetric group S_n of permutations of $\{1,2,\ldots,n\}$. Let Ω be the set of 2-element subsets of $\{1,2,\ldots,n\}$. There is a "natural" action of G on Ω given by $\{i,j\}g = \{ig, jg\}$. (You are not required to show that this is an action.) Prove the following assertions:

(a) If $n = 2$, the action is not faithful.

(b) If $n = 3$, the action is doubly transitive.

(c) If $n = 4$, the action is imprimitive.

(d) If $n \geq 5$, the action is primitive but not doubly transitive.

2 Let $\mathrm{Aut}(G)$ denote the automorphism group of the group G.

(a) Let V_4 denote the Klein group. Prove that $\mathrm{Aut}(V_4) \cong S_3$.

(b) Prove that $\mathrm{Aut}(S_3) \cong S_3$.

(c) Find another group G such that $\mathrm{Aut}(G) \cong G$.

(d) Let G be the elementary abelian group of order 8. Prove that $|\mathrm{Aut}(G)| = 168$. Is there any connection with the question on Problem Sheet 1?

3 Let G be a finite group of order greater than 2. Prove that G has a non-identity automorphism. (Hint: treat abelian and non-abelian groups separately.)

Remark: It is also true that, if G is an infinite group, then G has a non-identity automorphism; but the proof requires the Axiom of Choice. (Can you prove this?)