1 Let $H$ and $K$ be subgroups of a group $G$.

(a) Show that $H \cap K$ is a subgroup.

(b) Show that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|},$$

where $HK = \{hk : h \in H, k \in K\}$. [Hint: given $x \in HK$, in how many different ways can we write it in the form $hk$ with $h \in H$ and $k \in K$?]

(c) Show that, if $H$ is a normal subgroup of $G$, then $HK$ is a subgroup of $G$.

(d) Give an example of subgroups $H$ and $K$ for which $HK$ is not a subgroup.

2 Let $A$ be the group of all complex roots of unity, with the operation of multiplication. Let $Q$ be the group of rational numbers, with the operation of addition. Let $\theta : Q \to A$ be the map given by

$$q\theta = e^{2\pi i q}.$$ 

Prove that $\theta$ is a homomorphism, with image $A$ and kernel $\mathbb{Z}$. Hence show that $\mathbb{Q}/\mathbb{Z} \cong A$.

Is $A$ isomorphic to the infinite cyclic group $C_\infty$?

3 Verify the following table:

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Rotation group</th>
<th>Symmetry group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>$A_4$</td>
<td>$S_4$</td>
</tr>
<tr>
<td>Cube</td>
<td>$S_4$</td>
<td>$S_4 \times C_2$</td>
</tr>
</tbody>
</table>

(Here $C_n$ is the cyclic group of order $n$, $S_n$ and $A_n$ the symmetric and alternating groups of degree $n$.)
Let $n = a_0 + a_1 p + \cdots + a_r p^r$, where $p$ is prime and $0 \leq a_i \leq p-1$ for $i = 0, \ldots, r$, be the expression for $n$ in base $p$.

(a) Show that the symmetric group $S_n$ contains a subgroup which is the direct product of $a_i$ symmetric groups of degree $p^i$, for $i = 0, \ldots, r$.

(b) Show that a Sylow $p$-subgroup of $S_{p^i}$ has order $p^m$, where $m = 1 + p + \cdots + p^{i-1}$, and construct such a subgroup.

(c) Hence show that $S_n$ has a Sylow $p$-subgroup.

(d) Use Cayley’s Theorem to show that every finite group has a Sylow $p$-subgroup.