Queen Mary
University of London

MTHM024/MTH714U Group Theory

Problem Sheet 1 — 20 October 2011

The problem sheets in this course are for "formative assessment" only; there is no coursework component in the assessment.

Any work handed in by the lecture on the date at the top of the sheet will be marked and returned to you in the next week's lecture.

1 Let H and K be subgroups of a group G.

(a) Show that $H \cap K$ is a subgroup.

(b) Show that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|},$$

where $HK = \{hk : h \in H, k \in K\}$. [*Hint:* given $x \in HK$, in how many different ways can we write it in the form hk with $h \in H$ and $k \in K$?]

(c) Show that, if H is a normal subgroup of G, then HK is a subgroup of G.

(d) Give an example of subgroups H and K for which HK is not a subgroup.

2 Let A be the group of all complex roots of unity, with the operation of multiplication. Let $\mathbb{Q}$ be the group of rational numbers, with the operation of addition. Let $\theta : \mathbb{Q} \to A$ be the map given by

$$q\theta = \mathrm{e}^{2\pi \mathrm{i} q}.$$

Prove that θ is a homomorphism, with image A and kernel $\mathbb{Z}$. Hence show that $\mathbb{Q}/\mathbb{Z} \cong A$.

Is A isomorphic to the infinite cyclic group C_∞?

3 Verify the following table:

Polyhedron	Rotation group	Symmetry group
Tetrahedron	A_4	S_4
Cube	S_4	$S_4 \times C_2$

(Here C_n is the cyclic group of order n, S_n and A_n the symmetric and alternating groups of degree n.)

4 Let $n = a_0 + a_1 p + \cdots + a_r p^r$, where p is prime and $0 \leq a_i \leq p-1$ for $i = 0, \ldots, r$, be the expression for n in base p.

(a) Show that the symmetric group S_n contains a subgroup which is the direct product of a_i symmetric groups of degree p^i, for $i = 0, \ldots, r$.

(b) Show that a Sylow p-subgroup of S_{p^i} has order p^m, where $m = 1 + p + \cdots + p^{i-1}$, and construct such a subgroup.

(c) Hence show that S_n has a Sylow p-subgroup.

(d) Use Cayley's Theorem to show that every finite group has a Sylow p-subgroup.