

5 Periodic continued fractions

Continued fractions give a very useful way of representing real numbers if you are interested in finding good approximations; also we will soon see uses of them in solving diophantine equations. But for ordinary arithmetic, they are hopeless: there is no simple rule even for adding two continued fractions. Here is an example.

We have

$$\sqrt{2} + 1 = [2; 2, 2, 2, \dots] = [\overline{2}].$$

If we add it to itself, we get

$$2(\sqrt{2} + 1) = [4; 1, 4, 1, \dots] = [\overline{4; 1}].$$

Could you imagine a rule to produce this? Maybe. But adding again,

$$3(\sqrt{2} + 1) = [7; 4, 8, 4, 8, \dots] = [7; \overline{4, 8}],$$

and it is difficult to see how any rule would produce this.

But worse is to come. We have

$$\sqrt{5} + 2 = [4; 4, 4, 4, \dots] = [\overline{4}],$$

but adding this to $\sqrt{2} + 1$ we get

$$\sqrt{2} + \sqrt{5} + 3 = [6; 1, 1, 1, 6, 8, 1, 2, 1, 2, 2, \dots].$$

We see no pattern at all, and it really stretches the imagination to see how any simple rule could produce this from $[\overline{2}] + [\overline{4}]$.

In fact, we are going to show in this chapter that the continued fraction for $\sqrt{2} + \sqrt{5} + 3$ is non-periodic: it never repeats. We will decide just which irrational numbers have periodic continued fractions: they are precisely the quadratic numbers. Now convince yourself that $\sqrt{2} + \sqrt{5} + 3$ does not satisfy a quadratic with rational coefficients. (Its minimal polynomial has degree 4.)

Definition The infinite continued fraction

$$[a_0; a_1, a_2, \dots]$$

is *periodic* if there exist integers k, l with $k > 0$ such that

$$a_{n+k} = a_n \text{ for all } n \geq l.$$

It is *purely periodic* if there exists $k > 0$ such that

$$a_{n+k} = a_n \text{ for all } n \geq 0.$$

If $a_{n+k} = a_n$ for all $n \geq l$, we write the continued fraction as

$$[a_0; a_1, a_2, \dots, a_{l-1}, \overline{a_l, a_{l+1}, \dots, a_{l+k-1}}].$$

This is a similar notation to the one used for periodic decimals. For example,

$$\begin{aligned} [2; 1, 2, 1, 2, 1, 2, 1, \dots] &= [\overline{2; 1}] \\ [3; 5, 2, 1, 2, 1, 2, 1, \dots] &= [3; 5, \overline{2, 1}]. \end{aligned}$$

We now calculate these two continued fractions. Let $c = [\overline{2; 1}]$. Then

$$\begin{aligned} c &= [2; 1, c] \\ &= \frac{[2, 1, c]}{[1, c]} \text{ (by Proposition 2.3(b))} \\ &= \frac{3c + 2}{c + 1}. \end{aligned}$$

So $c^2 + c = 3c + 2$, so that $c^2 - 2c - 2 = 0$, or $c = 1 \pm \sqrt{3}$. But $c > 2$, so we must take the plus sign; $c = 1 + \sqrt{3}$.

Now let $d = [3; 5, \overline{2, 1}]$. Then

$$\begin{aligned} d &= [3; 5, c] \\ &= \frac{[3, 5, c]}{[5, c]} \\ &= \frac{16c + 3}{5c + 1} \\ &= \frac{19 + 16\sqrt{3}}{6 + 5\sqrt{3}} \\ &= \frac{(19 + 16\sqrt{3})(6 - 5\sqrt{3})}{(6 + 5\sqrt{3})(6 - 5\sqrt{3})} \\ &= \frac{126 - \sqrt{3}}{39}. \end{aligned}$$

Note that d , like c , is a “quadratic irrational”, an algebraic integer satisfying a quadratic equation. (We saw this already for c ; and d satisfies $(39x - 126)^2 = 3$.)

In this chapter we are going to show that the result suggested by these examples is true in general. A real number has a periodic continued fraction if and only if it is a quadratic irrational. We will also find which numbers have purely periodic continued fractions. We will apply these results to sums of squares and to a diophantine equation called *Pell’s equation* in the next chapter.

Recall from Chapter 2 that a *quadratic irrational* is a real number of the form $x = u + v\sqrt{d}$, where u and v are rational numbers, $v \neq 0$, and d is a squarefree integer greater than 1. (We require $v \neq 0$ and $d > 1$ since, if either of these conditions failed, x would be rational.) Quadratic irrationals are precisely the roots of irreducible quadratics with rational coefficients.

If $x = u + v\sqrt{d}$ is a quadratic irrational, we define its *conjugate* to be $x' = u - v\sqrt{d}$. Note that x and x' are the two roots of the same irreducible quadratic.

Now we define a *reduced quadratic irrational* to be a quadratic irrational x such that x and its conjugate x' satisfy

$$x > 1 \quad \text{and} \quad -1 < x' < 0.$$

In our worked example, $c = 1 + \sqrt{3} > 0$ and $-1 < c' = 1 - \sqrt{3} < 0$, so c is a reduced quadratic irrational. On the other hand, $d = (126 - \sqrt{3})/39 > 1$ but $d' = (126 + \sqrt{3})/39$ is greater than d . So d' is not reduced.

We are going to prove the following result:

Theorem 5.1 (a) *A real number has periodic continued fraction if and only if it is a quadratic irrational.*

(b) *A real number has purely periodic continued fraction if and only if it is a reduced quadratic irrational.*

There are four things to prove here, and the proofs will take the rest of this chapter.

Proof 1 We begin by showing that a number with a purely periodic continued fraction is a reduced quadratic irrational.

Let $x = [\overline{a_0; a_1, \dots, a_{k-1}}]$. We will suppose that $k \geq 3$; the argument for $k = 1, 2$ is easy to do directly, or we can simply pretend that the period is longer than it is (for example, $[\overline{2; 1}] = [\overline{2; 1, 2, 1}]$).

We know that x is irrational, since the continued fraction for a rational number terminates. Also, just as we argued for the number $c = [2; \overline{1}]$, we have

$$\begin{aligned} x &= [a_0; a_1, \dots, a_{k-1}, x] \\ &= \frac{xp_{k-1} + p_{k-2}}{xq_{k-1} + q_{k-2}}, \end{aligned}$$

where $c_i = p_i/q_i$ is the i th convergent of $[a_0; a_1, \dots, a_{k-1}]$.

Hence $x^2q_{k-1} + x(q_{k-2} - p_{k-1}) - p_{k-2} = 0$, so that y is a quadratic irrational.

Also, $a_0 = a_k \geq 1$ (remember that all terms except possibly the first in a continued fraction are positive), so $y > a_0 \geq 1$. It remains to show that the algebraic conjugate y' of y satisfies $-1 < y' < 0$.

From the properties of quadratic equations, we have $xy' = -p_{k-2}/q_{k-1} < 0$. Also,

$$-x' = \frac{p_{k-2}}{xq_{k-1}} < \frac{p_{k-1}}{xq_{k-1}} = \frac{c_{k-1}}{x},$$

because $p_{k-2} < p_{k-1}$; and also

$$-x' = \frac{p_{k-2}}{xq_{k-1}} < \frac{p_{k-2}}{xq_{k-2}} = \frac{c_{k-2}}{x},$$

because $q_{k-2} < q_{k-1}$. One of $k-1$ and $k-2$, say j , is even; and we know from Corollary 2.8 that $c_j < x$. So $-x' < 1$, or $x' > -1$. Now we have verified all parts of the definition of a reduced quadratic irrational.

Proof 2 Next we show that any number with periodic continued fraction is a quadratic irrational.

Let $y = [a_0; a_1, \dots, a_m, \overline{a_{m+1}, \dots, a_{m+k}}]$. Let $z = [\overline{a_{m+1}, \dots, a_{m+k}}]$. By Proof 1, z is a (reduced) quadratic irrational, say $z = u + v\sqrt{d}$, where u and v are rational numbers and d is a squarefree integer. We have

$$\begin{aligned} y &= [a_0; a_1, \dots, a_m, z] = \frac{[a_0, \dots, a_m, z]}{[a_1, \dots, a_m, z]} \\ &= \frac{[a_0, \dots, a_m]z + [a_0, \dots, a_{m-1}]}{[a_1, \dots, a_m]z + [a_1, \dots, a_{m-1}]}. \end{aligned}$$

Let $[a_0, \dots, a_m] = A$, $[a_0, \dots, a_{m-1}] = B$, $[a_1, \dots, a_m] = C$, $[a_1, \dots, a_{m-1}] = D$ (these are all positive integers). Then

$$\begin{aligned} y &= \frac{Az + B}{Cz + D} \\ &= \frac{Au + B + Av\sqrt{d}}{Cu + D + Cv\sqrt{d}} \\ &= \frac{(Au + B + Av\sqrt{d})(Cu + D - Cv\sqrt{d})}{(Cu + D)^2 - (Cv)^2d}, \end{aligned}$$

which is a quadratic irrational since it has the form $U + V\sqrt{d}$ for some rational numbers U and V .

Now our goal is to prove the converse of the last two results: if x is a [reduced] quadratic irrational, then its continued fraction is [purely] periodic. Let us begin with an example.

Example Find the continued fraction of $2 + \sqrt{7}$. Note that $2 + \sqrt{7}$ is reduced: it is greater than 1, and its algebraic conjugate $2 - \sqrt{7}$ lies between -1 and 0 .

$$\begin{aligned} x_0 &= 2 + \sqrt{7}, \\ a_0 &= \lfloor 2 + \sqrt{7} \rfloor = 4 & x_1 &= 1/(2 + \sqrt{7} - 4) = (2 + \sqrt{7})/3, \\ a_1 &= \lfloor (2 + \sqrt{7})/3 \rfloor = 1 & x_2 &= 3/(2 + \sqrt{7} - 3) = (1 + \sqrt{7})/2, \\ a_2 &= \lfloor (1 + \sqrt{7})/2 \rfloor = 1 & x_3 &= 2/(1 + \sqrt{7} - 2) = (1 + \sqrt{7})/3 \\ a_3 &= \lfloor (1 + \sqrt{7})/3 \rfloor = 1 & x_4 &= 3/(1 + \sqrt{7} - 3) = 2 + \sqrt{7} = x_0 \end{aligned}$$

So

$$2 + \sqrt{7} = [4; 1, 1, 1].$$

Note that all of x_0, x_1, x_2, x_3 are reduced quadratic irrationals, and we can read off their continued fractions: for example, $x_2 = [1; 1, 4, 1]$. Other observations which will be important in the proof are that, in each case, $x_i = (P_i + \sqrt{7})/Q_i$, where P_i and Q_i are integers (the P_i are $2, 2, 1, 1, \dots$ and the Q_i are $1, 3, 2, 3, \dots$); and $0 < P_i < \sqrt{7}$, $0 < Q_i < 2\sqrt{7}$.

We will see that all these properties hold quite generally.

Before we start the proofs, we introduce a slightly different way of writing quadratic irrationals.

Lemma 5.2 (a) *A real quadratic irrational can be written as $x = (P + \sqrt{D})/Q$, where P, Q are integers, D is a positive integer which is not a square, and Q divides $D - P^2$.*

(b) *If x is reduced, then $0 < P < \sqrt{D}$ and $0 < Q < 2\sqrt{D}$.*

Proof (a) We know that $x = u + v\sqrt{d}$ where u and v are rationals and d is squarefree.

Suppose first that v is positive. Let q be the least common multiple of the denominators of u and v , and $u = p/q$, $v = r/q$. Then

$$x = \frac{p + r\sqrt{d}}{q} = \frac{p + \sqrt{r^2d}}{q} = \frac{pq + \sqrt{q^2r^2d}}{q^2}.$$

Put $P = pq$, $Q = q^2$, and $D = q^2r^2d$, and note that Q divides $P^2 - D$.

If $v < 0$, then write $-x$ in the specified form and then replace Q by $-Q$.

(b) Now suppose that x is reduced; recall that this means $x > 1$ and $-1 < x' < 0$, where x' is the conjugate of x (so $x' = (P - \sqrt{D})/Q$). Then

- $x > 0 > x'$, so $(P + \sqrt{D})/Q > (P - \sqrt{D})/Q$. Hence $Q > 0$.
- $x > 1 > -x'$, so $(P + \sqrt{D})/Q > (-P + \sqrt{D})/Q$. Hence $P > 0$.
- $x' < 0$, so $P - \sqrt{D} < 0$. Hence $P < \sqrt{D}$.
- $x > 1$, so $(P + \sqrt{D})/Q > 1$. Hence $Q < P + \sqrt{D} < 2\sqrt{D}$.

Proof 3 We show that a reduced quadratic irrational has a purely periodic continued fraction.

The proof is quite long, so I begin by outlining the steps. Let x be a reduced quadratic irrational.

- We show that, if we run one step of the continued fraction algorithm, that is, with $x_0 = 0$ and $a_0 = \lfloor x_0 \rfloor$, $x_1 = 1/(x_0 - a_0)$, that x_1 is also a reduced quadratic irrational.
- It follows that all the numbers x_n that come up in the algorithm are reduced quadratic irrationals.
- So all these numbers x_n have the form $(P_n + \sqrt{D})/Q_n$, where $0 < P_n < \sqrt{D}$ and $0 < Q_n < 2\sqrt{D}$.
- Since there are only finitely many possibilities for P_n and Q_n , we must eventually return to a value we saw before; the algorithm is periodic from this point.
- Finally we show it is periodic right from the start, that is, purely periodic.

Now for the details!

Suppose that x is a reduced quadratic irrational. We start building the continued fraction for $x_0 = x$. The first step is

$$a_0 = \lfloor x_0 \rfloor, \quad x_1 = \frac{1}{x_0 - a_0}.$$

Claim: x_1 is reduced.

Certainly $x_1 > 1$, since $x_1 = 1/(x_0 - a_0)$ and $x_0 - a_0 < 1$. We have to show that $-1 < x'_1 < 0$.

Let $P^* = Qa_0 - P$. Then

$$\begin{aligned}
x_1 &= \frac{1}{(P + \sqrt{D})/Q - a_0} \\
&= \frac{1}{(P + \sqrt{D} - Qa_0)/Q} \\
&= \frac{1}{(-P^* + \sqrt{D})/Q} \\
&= \frac{P^* + \sqrt{D}}{(D - (P^*)^2)/Q} \\
&= \frac{P^* + \sqrt{D}}{Q^*},
\end{aligned}$$

where $Q^* = (D - (P^*)^2)/Q$. Now

$$Q^* = \frac{D - (Qa_0 - P)^2}{Q} = -Qa_0^2 + 2Pa_0 + \frac{D - P^2}{Q},$$

so Q^* is an integer. Moreover, Q^* divides $D - (P^*)^2$ (the quotient is just Q). So we have written y_1 in the same form as y , with the same D , but maybe different P and Q .

Also, we have $x_1 = 1/(x - a_0)$, and so $x'_1 = 1/(x' - a_0)$. Now $x' < 0$, so $x' - a_0 < -a_0 \leq -1$; so $-1 < x'_1 < 0$, as required.

Our claim is proved.

Now calculate the continued fraction expansion of x . In the general step, we start with $x_n = (P_n + \sqrt{D})/Q_n$, where Q_n divides $P_n^2 - D$; assume inductively that x_n is reduced, so $0 < P_n < \sqrt{D}$, $0 < Q_n < 2\sqrt{D}$. Then we put

$$a_n = \lfloor x_n \rfloor, \quad x_{n+1} = \frac{1}{x_n - a_n}.$$

By the above argument, we have $y_{n+1} = (P_{n+1} + \sqrt{D})/Q_{n+1}$, where the same conditions hold.

Now P_n and Q_n are integers satisfying $0 < P_n < \sqrt{D}$ and $0 < Q_n < 2\sqrt{D}$. There are only finitely many possible values of P_n and Q_n , so after some number of steps, we must return to values we have seen before. Suppose this first happens when $x_n = x_m$. Clearly the sequence repeats after this point, that is, $x_{i+k} = x_i$ for all $i \geq m$, where $k = n - m$.

We have to show that the repetition starts with $m = 0$. If not, then we have $x_{m-1} \neq x_{n-1}$. But

$$x_m = \frac{1}{x_{m-1} - a_{m-1}} = x_n = \frac{1}{x_{n-1} - a_{n-1}},$$

so

$$x'_m = \frac{1}{x'_{m-1} - a_{m-1}} = x'_n = \frac{1}{x'_{n-1} - a_{n-1}},$$

So $x'_{m-1} - a_{m-1} = x'_{n-1} - a_{n-1}$. Thus x'_{m-1} and x'_{n-1} differ by an integer. But they both lie between -1 and 0 , so they are equal, whence $x_{m-1} = x_{n-1}$, contrary to assumption: we assumed that x_n was the first repeat!

So, finally, we have proved the converse implication in part (b) of the big theorem: that is, the irrational number y has a purely periodic continued fraction if and only if it is a reduced quadratic irrational.

Proof 4 The final part is to prove part (a) of the theorem: the irrational number x has a periodic continued fraction if and only if it is a quadratic irrational.

We have proved the forward implication in Proof 2. So suppose that x is a quadratic irrational; we have to show that its continued fraction is periodic (from some point on).

Calculate its continued fraction: that is, put $x_0 = x$, and then

$$a_n = \lfloor x_n \rfloor, \quad x_{n+1} = \frac{1}{x_n - a_n}$$

for $n \geq 0$. It is clear that all the x_n are quadratic irrationals, since subtracting an integer from a quadratic irrational, and taking the reciprocal of one, gives again a quadratic irrational. We have to prove that, for some value of n , the number x_n is a reduced quadratic irrational. Then by part (b) of the Theorem, the continued fraction is periodic from that point on. So we have to prove that, if n is large enough, then $x_n > 1$ and $-1 < x'_n < 0$.

By construction, we have $x_n > 1$ for all $n > 0$, since $x_n = 1/(x_{n-1} - a_{n-1})$ and $x_{n-1} - a_{n-1} < 1$. So this part is easy!

Also, we have for any $n > 0$

$$x = [a_0; a_1, \dots, a_{n-1}, x_n] = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}},$$

by standard results about continued fractions.

Hence

$$x' = \frac{x'_n p_{n-1} + p_{n-2}}{x'_n q_{n-1} + q_{n-2}},$$

so rearranging we obtain

$$x'_n = \frac{-x' q_{n-2} + p_{n-2}}{x' q_{n-1} - p_{n-1}}.$$

(It is your job to rearrange the preceding equation to make x'_n the subject; you will need an earlier result connecting the ps and qs .)

So, taking factors q_{n-2} and q_{n-1} out of numerator and denominator,

$$x'_n = -\frac{q_{n-2}}{q_{n-1}} \frac{x' - c_{n-2}}{y' - c_{n-1}},$$

where c_n is the n th convergent. (Remember that $c_n = p_n/q_n$.)

Since $c_n \rightarrow x$ as $n \rightarrow \infty$, we have

$$-\frac{q_{n-1}}{q_{n-2}} x'_n = \frac{x' - c_{n-2}}{x' - c_{n-1}} \rightarrow \frac{x' - x}{x' - x} = 1$$

as $n \rightarrow \infty$.

So we can choose N large enough that, for $n > N$, we have $|(x' - c_{n-2})/(x' - c_{n-1}) - 1| < 1$, and so this fraction is positive. Thus $-(q_{n-1}/q_{n-2})x'_n > 0$, so that $x'_n < 0$.

Also, we can ensure that n is also large enough that $|c_n - x| < |x' - x|$ for $n > N$.

If $x' < x$, we use the fact that even-numbered convergents are smaller than x and odd-numbered convergents are greater; choosing n even, we have $x' < c_{n-2} < x < c_{n-1}$. If $x < x'$, then choosing n odd we have $c_{n-1} < x < c_{n-2} < x'$. In either case, we have

$$-x'_n = \frac{q_{n-2}}{q_{n-1}} \left| \frac{x' - c_{n-2}}{x' - c_{n-1}} \right| < 1,$$

so finally we conclude

$$-1 < x'_n < 0$$

and x_n is a reduced quadratic irrational, as required.

The proof is complete.