Systèmes dynamiques/*Dynamical systems*

Directional Entropy of Rotation Sets

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Abstract.- For (X,T) a mixing subshift of finite type, and $f : X \to \mathbb{R}^d$ of summable variation, we give a characterisation of directional entropy (see [4]) at interior and exposed points of the corresponding generalised rotation set.

Entropie Directionnelle des Ensembles de Rotation

Résumé.- Pour (X,T) un sous-shift de type fini mélangeant, et $f : X \to \mathbb{R}^d$ une fonction de variation sommable, on caractérise l'entropie directionnelle (voir [4]) aux points intérieur à l'ensemble de rotation, et aux points exposés de l'ensemble de rotation.

Version Française Abrégée - Soit (X,T) un système dynamique donné par une surjection continue $T : X \to X$ où X est compact et métrisable. Soit $f_1, \ldots, f_d$ les fonctions coordonnées de la fonction continue $f : X \to \mathbb{R}^d$. Si $\mathcal{M}$ est l'ensemble des mesures de probabilité boréliennes qui sont invariantes par T, alors f permet de définir l'application $f_* : \mathcal{M} \to \mathbb{R}^d$:

$$f_*(\mu) = \left(\int f_1 \, d\mu, \ldots, \int f_d \, d\mu \right).$$

On appelle $f_*(\mu)$ le *vecteur de rotation* de la mesure $\mu \in \mathcal{M}$. L'image $f_*(\mathcal{M})$, appelée *ensemble de rotation*, est compacte et convexe. La terminologie est motivée par le cas classique des vecteurs de rotation d'une application du tore (voir [9], [13]). Ici l'on définie $f = F\pi^{-1} - \pi^{-1}$, où F est un relèvement de l'application au revêtement universel $\mathbb{R}^d$, et π^{-1} est une branche inverse du revêtement.

D'autres exemples intéressants d'ensembles de rotation sont les suivants : la boule unité pour la norme stable de l'espace d'homologie réelle de certains surfaces [11], [12]; l'ensemble des barycentres $\int_{S^1} z \, d\mu(z)$ des mesures de probabilité invariantes sous une application du cercle [1], [5], [6], [7]; le spectre des exposants de Lyapunov pour une application unidimensionnelle [2].

Geller & Misiurewicz [4] ont récemment introduit les notions de mesures ergodiques *directionnelles* et *perdues*. Une mesure ergodique est appelée *directionnelle* si toutes les mesures ayant leur support contenu dans le support de μ ont même vecteur de rotation que μ. Sinon, une mesure ergodique μ est appellée *perdue*. L'*entropie directionnelle* $\mathcal{H}(\varrho)$ d'un vecteur de rotation ϱ est ainsi définie comme supremum des entropies des mesures directionnelles dans $f_*^{-1}(\varrho)$, (où on définit le supremum de l'ensemble vide comme nul). On définit aussi $H(\varrho)$ comme le supremum des entropies sur *toutes* les mesures dans $f_*^{-1}(\varrho)$.

Geller & Misiurewicz [4] considèrent en détail le cas d'un sous-shift de type fini mélangeant, où la fonction f est constante sur les cylindres de longueur un. Fan & Feng [3] ont traité le cas plus général où f est n'importe quelle fonction localement constante.

Pour énoncer nos résultats, il faut d'abord se rappeler de quelques notions du formalisme thermodynamique. La *pression* d'une fonction continue $g : X \to \mathbb{R}$ est définie par $P(g) = \sup_{\mu\in\mathcal{M}} \left(h(\mu) + \int g\, d\mu\right)$, et toute mesure $m \in \mathcal{M}$ qui atteint ce suprémum est appelée un *état d'équilibre* pour g. Pour $v \in \mathbb{R}^d$, soit $v.f$ la fonction $v_1 f_1 + \ldots + v_d f_d$. On définit $p : \mathbb{R}^d \to \mathbb{R}$ par $p(v) = P(v.f)$. Soit $\mathcal{M}_f(v)$ l'ensemble des états d'équilibre de $v.f$. Soit $\partial p(v)$ la *sous-dérivée* de p au point v (voir [15]).

Théorème 1. *Soit (X,T) un système dynamique pour lequel l'application d'entropie $h : \mathcal{M} \to \mathbb{R}$ est semi-continue supérieurement, et soit $f : X \to \mathbb{R}^d$ continue. Alors*
(a) $f_*(\mathcal{M}) = \overline{f_*(\mathcal{M}_f(\mathbb{R}^d))} = \overline{\partial p(\mathbb{R}^d)}$.
Si ϱ appartient à l'intérieur rélatif $ri(f_*(\mathcal{M}))$, alors
(b) il existe $v \in \mathbb{R}^d$ tel que $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v)$, et
(c) $m \in f_*^{-1}(\varrho)$ satisfait $h(m) = H(\varrho)$ si et seulement si $m \in f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d)$.

Notez que h est semi-continue supérieurement si, par exemple, T est une application C^∞ d'une variété compacte, ou T est n'importe quel sous-shift symbolique.

Théorème 2. *Soit (X,T) un sous-shift de type fini mélangeant (unidirectionnel ou bidirectionnel). Soit f de variation sommable, où les fonctions coordonnées f_i sont cohomologiquement indépendantes. Les fonctions H et $\mathcal{H}$ coïncident dans l'intérieur de $f_*(\mathcal{M})$, et sur l'ensemble des points exposés. Pour tout point ϱ de l'intérieur, $\mathcal{H}(\varrho) = H(\varrho)$ est atteint par une seule mesure dans $f_*^{-1}(\varrho)$, et cette mesure est perdue. Pour les points ϱ exposés, $\mathcal{H}(\varrho) = H(\varrho)$ est atteint par au moins une mesure directionnelle dans $f_*^{-1}(\varrho)$, et par aucune mesure perdue.*

La partie la plus importante du théorème 2, ainsi que le résultat principal de cette Note, est le fait que les deux entropies H et $\mathcal{H}$ sont (essentiellement) égales.

On se référa à [14] pour la définition d'un sous-shift de type fini mélangeant. On se rappelle que *variation sommable* veut dire que $\sum_{n=1}^\infty \mathrm{var}_n(f) < \infty$, où

$$\mathrm{var}_n(f) := \max\{|f(x)-f(y)| : x, y \text{ appartiennent au même cylindre de longueur } n\}.$$

Indépendance cohomologique signifie que toute combinaison linéaire non-triviale des f_i n'est *pas* égale à $\psi \circ T - \psi + c$ pour ψ Borel-mesurable, et $c \in \mathbb{R}$. Un vecteur de rotation ϱ est dit *exposé* si $\{\varrho\}$ est une face de $f_*(\mathcal{M})$. Un point exposé est en particulier un point extrême (voir [15]). Pour un ensemble *strictement* convexe, tout point sur le bord est un point exposé, donc on a

Corollaire. *Soit (X,T) un sous-shift de type fini mélangeant (unidirectionnel ou bidirectionnel). Soit f de variation sommable, où les fonctions coordonnées f_i sont cohomologiquement indépendantes. Si l'ensemble de rotation $f_*(\mathcal{M})$ est strictement convexe, alors les fonctions $\mathcal{H}$ et H coïncident.*

On trouvera plus de détails sur les résultats de cette Note, ainsi que des références plus larges, dans [8]. Je remercie Thierry Bousch, Ai Hua Fan, et Evgeny Verbitski pour leurs remarques pertinentes sur ce travail.

1. Introduction

Consider a dynamical system (X,T), given by a continuous surjection $T : X \to X$ of a compact metrizable space X. Suppose the continuous function $f : X \to \mathbb{R}^d$ has coordinate functions $f_1, \ldots, f_d$. If $\mathcal{M}$ denotes the set of T-invariant Borel probability measures, then f induces the map $f_* : \mathcal{M} \to \mathbb{R}^d$ given by

$$f_*(\mu) = \left(\int f_1 \, d\mu, \dots , \int f_d \, d\mu \right).$$

We call $f_*(\mu)$ the *rotation vector* of the measure $\mu \in \mathcal{M}$. The image $f_*(\mathcal{M})$, called the *rotation set*, is compact and convex.

Rotation sets arise in various contexts: rotation sets of torus maps [9], [13]; norm balls for the stable norm on the real homology space of various surfaces [11], [12]; the set of barycentres $\int_{S^1} z \, d\mu(z)$ of invariant probability measures for a circle map [1], [5], [6], [7]; the spectrum of Lyapunov exponents for one-dimensional maps [2].

Geller & Misiurewicz [4] call an ergodic measure μ *directional* if all invariant measures whose support is contained in the support of μ have the same rotation vector as μ. Otherwise an ergodic μ is called *lost.* The *directional entropy* $\mathcal{H}(\varrho)$ of a rotation vector ϱ is then defined as the supremum of the entropies of those directional measures in $f_*^{-1}(\varrho)$, (where the supremum of the empty set is defined to be zero). Define also $H(\varrho)$ to be the supremum of the entropies over *all* measures in $f_*^{-1}(\varrho)$. Clearly $\mathcal{H}(\varrho) \le H(\varrho)$ for all ϱ.

Geller & Misiurewicz [4] consider in detail the case of a mixing subshift of finite type, where the function f is constant on length-one cylinder sets. Fan & Feng [3] consider the more general case where f is any locally constant function.

In order to state our results we first recall some thermodynamic ideas. The *pressure* of a continuous function $g : X \to \mathbb{R}$ is given by $P(g) = \sup_{\mu \in \mathcal{M}} \left(h(\mu) + \int g \, d\mu \right)$, and any $m \in \mathcal{M}$ realising this supremum is called an *equilibrium state* for g. For $v \in \mathbb{R}^d$, let $v.f$ denote the function $v_1 f_1 + \ldots + v_d f_d$, and define $p : \mathbb{R}^d \to \mathbb{R}$ by $p(v) = P(v.f)$. Let $\mathcal{M}_f(v)$ denote the set of equilibrium states of $v.f$. Let $\partial p(v)$ denote the *subdifferential* of p at v (see [15]).

Theorem 1. *Suppose (X,T) is a dynamical system with upper semi-continuous entropy map $h : \mathcal{M} \to \mathbb{R}$, and $f : X \to \mathbb{R}^d$ is continuous. Then*
(a) $f_*(\mathcal{M}) = \overline{f_*(\mathcal{M}_f(\mathbb{R}^d))} = \overline{\partial p(\mathbb{R}^d)}$.
If the rotation vector ϱ lies in the relative interior $ri(f_*(\mathcal{M}))$, then
(b) there exists $v \in \mathbb{R}^d$ with $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v)$, and
(c) $m \in f_*^{-1}(\varrho)$ satisfies $h(m) = H(\varrho)$ if and only if $m \in f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d)$.

Note that upper semi-continuity of h is guaranteed, for example, if T is a C^∞ map of a compact manifold, or if T is any symbolic subshift.

Theorem 2. *Let (X,T) be a (one-sided or two-sided) mixing subshift of finite type. Let $f : X \to \mathbb{R}^d$ be of summable variation, where its coordinate functions f_i are cohomologically independent. Then $\mathcal{H}(\varrho) = H(\varrho)$ at all interior and exposed points ϱ of the rotation set $f_*(\mathcal{M})$. If ϱ lies in the interior of $f_*(\mathcal{M})$, then $\mathcal{H}(\varrho) = H(\varrho)$ is attained by a unique measure in $f_*^{-1}(\varrho)$, and this measure is lost. If ϱ is exposed, then $\mathcal{H}(\varrho) = H(\varrho)$ is attained by at least one directional measure in $f_*^{-1}(\varrho)$, and is not attained by any lost measures.*

See [14] for the definition of a mixing subshift of finite type. Recall that summable variation means $\sum_{n=1}^\infty \mathrm{var}_n(f) < \infty$, where

$$\mathrm{var}_n(f) := \max\{|f(x) - f(y)| : x, y \text{ belong to the same length-}n\text{ cylinder}\}.$$

Cohomological independence means that each non-trivial linear combination of the f_i *cannot* equal $\psi \circ T - \psi + c$ (an essential coboundary) for Borel measurable ψ

and $c \in \mathbb{R}$. It ensures $f_*(\mathcal{M})$ has interior. A rotation vector ϱ is called *exposed* if $\{\varrho\}$ is a face of $f_*(\mathcal{M})$. Exposed points are in particular extremal points (see [15]). For a *strictly* convex set, every point on the boundary is an exposed point, so we have

Corollary. *Let* (X,T) *and* $f : X \to \mathbb{R}^d$ *be as in Theorem 2. If the rotation set* $f_*(\mathcal{M})$ *is strictly convex, then the functions* $\mathcal{H}$ *and* H *coincide.*

2. Proof of Theorem 1

(a) Clearly $\overline{f_*(\mathcal{M}_f(\mathbb{R}^d))} \subset f_*(\mathcal{M})$, since $\mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}$. Compactness of $f_*(\mathcal{M})$ means in fact $\overline{f_*(\mathcal{M}_f(\mathbb{R}^d))} \subset f_*(\mathcal{M})$.

We can show (see Prop. 1, [8]) that $\partial p(v) = f_*(\mathcal{M}_f(v))$ for all $v \in \mathbb{R}^d$. The ingredients for this are the representation of an equilibrium state as a tangent functional to pressure, the upper semi-continuity of h, and a little convex geometry. We therefore have $\partial p(\mathbb{R}^d) = f_*(\mathcal{M}_f(\mathbb{R}^d))$, and hence $\overline{\partial p(\mathbb{R}^d)} = \overline{f_*(\mathcal{M}_f(\mathbb{R}^d))}$.

To show that $f_*(\mathcal{M}) \subset \overline{\partial p(\mathbb{R}^d)}$, let $\mu \in \mathcal{M}$ be arbitrary. Defining

$$A_\mu(v_1, \ldots, v_d) = h(\mu) + v_1 \int f_1 \, d\mu + \ldots + v_d \int f_d \, d\mu = h(\mu) + v.f_*(\mu),$$

the definition of pressure implies that $p(v) \geq A_\mu(v)$ for all $v \in \mathbb{R}^d$. Thus $Graph(p)$ lies above the hyperplane $Graph(A_\mu)$ in $\mathbb{R}^{d+1}$, possibly touching it tangentially.

This implies that the gradient of the hyperplane is contained in the closure of all subdifferentials of p, since if not then the graphs would intersect transversally. That is, the vector $f_*(\mu)$ belongs to the closure of $\partial p(\mathbb{R}^d)$. Since $\mu \in \mathcal{M}$ was arbitrary then $f_*(\mathcal{M}) \subset \overline{\partial p(\mathbb{R}^d)}$.

(b) Note that $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d)$ is non-empty by the above. Suppose (b) is false. Then there exists $v \neq v'$, and a measure m which belongs to $\mathcal{M}_f(v)$ but not to $\mathcal{M}_f(v')$, and a measure m' which belongs to $\mathcal{M}_f(v')$ but not to $\mathcal{M}_f(v)$, but such that $m, m' \in f_*^{-1}(\varrho)$. This implies that $\int v.f \, dm = \int v.f \, dm'$, but the fact that m is an equilibrium state for $v.f$, yet m' is not, gives $h(m) > h(m')$. Similarly we have $\int v'.f \, dm = \int v'.f \, dm'$, and now the fact that m' is an equilibrium state for $v'.f$, yet m is not, gives $h(m') > h(m)$. This is a contradiction.

(c) Let $m \in f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d)$, and let $v \in \mathbb{R}^d$ be such that $m \in \mathcal{M}_f(v)$. Now let μ be some measure in $f_*^{-1}(\varrho)$ which does not belong to the family $\mathcal{M}_f(\mathbb{R}^d)$. Then $\int v.f \, dm = \int v.f \, d\mu$, but the fact that m is an equilibrium state for $v.f$, whereas μ is not, gives $h(m) > h(\mu)$.

3. Proof of Theorem 2

For (X,T) and f as in the statement of Theorem 2, the function p is *strictly* convex, so that $\partial p(\mathbb{R}^d) = f_*(\mathcal{M}_f(\mathbb{R}^d))$ is both open in $\mathbb{R}^d$ and convex. Taking interiors in Theorem 1 gives that $int(f_*(\mathcal{M})) = f_*(\mathcal{M}_f(\mathbb{R}^d))$.

First let $\varrho \in int(f_*(\mathcal{M}))$. We claim there exists $m \in f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d)$ with $h(m) = H(\varrho)$, and that m is the unique measure in $f_*^{-1}(\varrho)$ with this property. By Theorem 1 we know there exists $v \in \mathbb{R}^d$ with $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v)$, Now suppose $v' \neq v$ satisfies $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v')$. Since the f_i are cohomologically independent then $v.f$ and $v'.f$ are not essentially cohomologous, so the equilibrium states $m_{v.f}, m_{v'.f}$ are distinct [16], Cor. 3.3(ii). So the argument used to prove Theorem 1 (b) again provides a contradiction. Thus there exists a unique $v \in \mathbb{R}^d$

for which $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(v)$. But $v.f$ is of summable variation, so it has a unique equilibrium state m, [16], Cor. 3.3(i). That is, $f_*^{-1}(\varrho) \cap \mathcal{M}_f(\mathbb{R}^d) = \{m\}$. The fact that m is the unique measure in $f_*^{-1}(\varrho)$ satisfying $h(m) = H(\varrho)$ then follows immediately from Theorem 1 (c).

The equilibrium state m is fully supported ([16], Cor. 3.3(i)). It follows that m is lost, since if not then $f_*(\mathcal{M})$ would be the singleton $\{\varrho\}$. But this cannot be, since a theorem of Livsic [10] (the proof in [14], Prop. 3.7 is easily adapted to the case of functions of summable variation) would imply that each coordinate function f_i is an essential coboundary, a contradiction.

It remains to show that $\mathcal{H}(\varrho) = H(\varrho)$. For any $0 < \alpha < H(\varrho)$ we will construct a subshift $Y_\alpha \subset X$, with topological entropy $h_{top}(Y_\alpha) > H(\varrho) - \alpha$, and such that

$$\sup_{y \in Y_\alpha} \left| \sum_{i=0}^{n-1} f(T^i y) - n\varrho \right| = O(1) \text{ as } n \to \infty. \tag{1}$$

Consequently any ergodic measure supported on Y_α will be directional, and in the fibre $f_*^{-1}(\varrho)$. Moreover, the variational principle (see Walters [17], [18]) means we can find an ergodic measure μ_α, with $supp(\mu_\alpha) \subset Y_\alpha$, such that $h(\mu_\alpha) > H(\varrho) - \alpha$.

By a *periodic block* of length n we mean an n-tuple of symbols $b = (b^{(1)}, \ldots, b^{(n)})$, such that $A(b^{(i)}, b^{(i+1)}) = 1$ for $i = 1, \ldots, n-1$, and $A(b^{(n)}, b^{(1)}) = 1$, where A is the transition matrix defining our subshift of finite type X. Each periodic block naturally corresponds to a unique measure $\mu \in \mathcal{M}$, so we define $f_*(b) = f_*(\mu)$, and call this the *rotation vector of the block b*.

Fix some $q \in \mathbb{N}$ such that $q > \frac{2(H(\varrho)-\alpha)}{\alpha}$. Choose $r > 0$ such that $B_{3rq}(\varrho) \subset f_*(\mathcal{M})$, (where $B_t(\varrho)$ denotes the open Euclidean ball of radius t around the point ϱ).

Fix $\varepsilon > 0$ such that $\varepsilon < rq$. We can also choose a natural number N satisfying the following four conditions:

(a) N is an integer multiple of q,

(b) $V(f)/N < r - \varepsilon/q$,

(c) There is a collection C of periodic blocks, each of length $M := N/q$, each beginning with the same symbol a, whose rotation vectors are ε-dense in $f_*(\mathcal{M})$,

(d) There is a collection B of length-N periodic blocks, of cardinality $|B| \geq e^{(H(\varrho)-\alpha/2)N}$, such that each $b \in B$ begins with the symbol a and satisfies $f_*(b) \in B_r(\varrho)$.

To each $\underline{b} = (b_i) \in B^{\mathbb{N}}$ we will identify a collection $Z(\underline{b})$ of points $x \in X$ of the form $x = b_1 c_1 b_2 c_2 \ldots$, where $c_i \in C$. Given $\underline{b}$, the j^{th} block c_j will depend on the $b_i, i \leq j$, and $c_i, i < j$, and will be chosen to maintain $|f^n(x) - n\varrho| = O(1)$ (rather than the $|f^n(x) - n\varrho| = O(n)$, which would occur if x were constructed solely from blocks in B). We think of the blocks $c \in C$ as being *corrective blocks*, since the c_i are chosen so as to reduce the distances $|f^n(x) - n\varrho|$. For any $\underline{b} \in B^{\mathbb{N}}$ we can show there exists at least one x of the above form, so that $Z(\underline{b})$ is non-empty. We then define Z to be the union of the $Z(\underline{b})$, and define $Y_\alpha = \overline{\bigcup_{i \geq 0} T^i(Z)}$. In [8] we show that (1) holds, and that $h_{top}(Y_\alpha) > H(\varrho) - \alpha$.

Now let ϱ be an exposed point of $f_*(\mathcal{M})$. So there exists some normal $v \in \mathbb{R}^d$ such that the supporting hyperplane $\{w \in \mathbb{R}^d : w.v = w.\varrho\}$ intersects $f_*(\mathcal{M})$ at the single point ϱ. Thus $\mu \in f_*^{-1}(\varrho)$ if and only if μ is $v.f$-optimal (for a continuous $g : X \to \mathbb{R}$, we say $\mu \in \mathcal{M}$ is *g-optimal* if $\int g\, d\mu = \sup_{m \in \mathcal{M}} \int g\, dm$). But then a theorem of Bousch (see [1], [8]) implies that any $m \in \mathcal{M}$ with $supp(m) \subset supp(\mu)$

is also $v.f$-optimal, and hence that $f_*(m) = f_*(\mu)$. So if our $\mu \in f_*^{-1}(\varrho)$ is ergodic then it is directional.

In fact $f_*^{-1}(\varrho)$ must contain at least one ergodic (and hence directional) measure, by the following argument. Since ϱ is exposed then it is also extremal in $f_*(\mathcal{M})$. Let $\nu \in \mathcal{M}$ be an extremal point of the convex set $f_*^{-1}(\varrho)$. Suppose ν is not ergodic. Then there exist distinct measures $\nu_1, \nu_2 \in \mathcal{M}$ and $\alpha \in (0,1)$ such that

$$\nu = \alpha\nu_1 + (1-\alpha)\nu_2. \tag{2}$$

Thus $f_*(\nu) = \alpha f_*(\nu_1) + (1-\alpha) f_*(\nu_2)$, and extremality of $\varrho = f_*(\nu)$ implies $f_*(\nu_1) = \varrho = f_*(\nu_2)$. But now since $\nu_1, \nu_2 \in f_*^{-1}(\varrho)$, (2) implies ν is not extremal in $f_*^{-1}(\varrho)$, a contradiction. Thus ν must be ergodic.

Now $f_*^{-1}(\varrho)$ is closed, and hence compact, so that there exists $m \in f_*^{-1}(\varrho)$ with $h(m) = H(\varrho)$. But extremality of ϱ means that almost every measure in the ergodic decomposition of m must belong to $f_*^{-1}(\varrho)$, so since entropy h is affine then there is some ergodic (and hence directional) $\mu \in f_*^{-1}(\varrho)$ with $h(\mu) \geq h(m) = H(\varrho)$. Thus $\mathcal{H}(\varrho) \geq H(\varrho)$. $\square$.

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