# CALCULATING HAUSDORFF DIMENSION OF JULIA SETS AND KLEINIAN LIMIT SETS

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Abstract. We present a new algorithm for efficiently computing the Hausdorff dimension of sets $X$ invariant under conformal expanding dynamical systems. By locating the periodic points of period up to $N$, we construct approximations $s_N$ which converge to $\dim(X)$ super-exponentially fast in $N$. This method can be used to give rigorous estimates for important examples, including hyperbolic Julia sets and limit sets of Schottky and quasifuchsian groups.

## 0. Introduction

Hausdorff dimension is a fundamental invariant of bi-Lipschitz homeomorphism. The Hausdorff dimension of a set is in general neither a rational number nor easily expressible in terms of well-known or easily computable irrationals, so that effective use of this invariant relies on its accurate computation. In general, however, computation of Hausdorff dimension is a non-trivial problem.

Dynamical systems provide not only a fertile source of examples of fractal sets, but also techniques for studying them (see for example [Pe]). Conformal dynamics is a particularly rich source of examples of sets with non-integer Hausdorff dimension, notably Julia sets of rational maps and limit sets of Kleinian groups.

In this paper we present an algorithm for accurately computing the Hausdorff dimension of a large class of such fractals, namely those for which the underlying dynamics is *hyperbolic*. For such sets $X$, our algorithm relies on locating the *periodic points* of the corresponding dynamical system $T : X \to X$. We construct a sequence of functions $\Delta_N$ in terms of the (first) derivatives of $T$, evaluated at the periodic points of period up to $N$. The largest zeros $s_N$ of the functions $\Delta_N$ provide a sequence of approximations to the Hausdorff dimension $\dim(X)$ which converges super-exponentially in $N$.

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More precisely, if $X$ is contained in a $C^\omega$ manifold of dimension $d \in \mathbb{N}$, then there exist $0 < \delta < 1$ and $C > 0$ such that $|s_N - \dim(X)| \leq C\delta^{N^{1+1/d}}$. The constants $\delta$ and $C$ both depend on the particular set $X$. Roughly speaking, for a fixed $d$, the stronger the hyperbolicity of the dynamics, the faster the sequence $s_N$ will converge.

In the systems we consider, the number of points of period up to $N$ increases at an exponential rate. Therefore the functions $\Delta_N$ can be constructed in at worst exponential time[1], so that the periodic point algorithm has, at worst, time complexity $O\left(\exp\left(aN^{\frac{d}{1+d}}\right)\right)$, $a > 0$. In particular, for any $d$ the algorithm has *sub-exponential* complexity, whereas previous algorithms such as those of McMullen [McM3], Bodart & Zinsmeister [BZ], Widom, Bensimon, Kadanoff & Shenker [WBKS], Garnett [Gar]), and Saupe [Sau], have at best *exponential* complexity.

A feature of the periodic point algorithm is that it easily lends itself to *rigorous* Hausdorff dimension estimates, via explicit bounds on the constants $\delta$ and $C$. (Such rigorous estimates appear to be more difficult to justify for the alternative algorithms cited above). Moreover, the *empirical* dimension estimates, inferred by comparing consecutive zeros $s_N$ (see §7, §8), offer an accuracy beyond that of the rigorous bounds.

Our interest in these problems was motivated by the recent paper of McMullen [McM3], who gave a different algorithm (the *eigenvalue* algorithm) for computing the Hausdorff dimension for the same class of examples as we consider. The complexity of the periodic point algorithm is lower than that of [McM3], and for sufficiently hyperbolic systems our empirical dimension estimates bear this out (see the examples below, and in §7 , §8). For systems of weaker hyperbolicity, however, the constant $\delta$ is closer to 1 and the inertia arising from sub - $O(\delta^{N^{1+1/d}})$ asymptotics means that McMullen's algorithm is for practical purposes superior.

An attractive feature of the periodic point algorithm is that it is *canonical*. At no stage is it necessary to make (non-canonical) choices of coordinates or of Markov partitions. One simply locates periodic points of the underlying dynamical system. Consequently the algorithm is particularly easy to implement.

The proof of the algorithm's rapid convergence is based on ideas from thermodynamic formalism, where Bowen and Ruelle interpreted dimension as an implicit solution of a pressure equation, as well as Grothendieck's classical work on the Fredholm determinants of nuclear operators.

We first apply our algorithm to computing the dimension of limit sets of two classes of Kleinian group:

*1. Schottky groups.* Fix $2p$ disjoint closed discs $D_1, \dots, D_{2p}$ in the plane, and Möbius maps $g_1, \dots, g_p$ such that each $g_i$ maps the interior of $D_i$ to the exterior of $D_{p+i}$. The corresponding Schottky group is defined as the group generated by $g_1, \dots, g_p$. The associated limit set $\Lambda$ is a Cantor subset of the union of the interiors of the discs $D_1, \dots, D_{2p}$.

[1]It is not inconceivable that algorithms will be developed to construct the functions $\Delta_N$ in sub-exponential time, in which case the complexity of our algorithm will be reduced. At present this appears to be possible only for *piecewise affine* systems, where the limit $\Delta$ of the $\Delta_N$ is actually available in closed form; however this is of only academic interest, since for such systems the Hausdorff dimension is easily computable (see [Pe, Th. 13.3]) without recourse to any algorithm.

We define a map $T$ on this union by $T|_{int(D_i)} = g_i$ and $T|_{int(D_{p+i})} = g_i^{-1}$. A *reflection group* is a Schottky group with $D_i = D_{p+i}$ for all $i = 1, \ldots, p$.

*2. Quasifuchsian groups.* Such groups are isomorphic to the fundamental group of a compact Riemann surface, and are obtained by a quasiconformal deformation of a Fuchsian group (a Kleinian group whose limit set is contained in some circle). The limit set $\Lambda$ of a quasifuchsian group is a simple closed curve. We can associate an expanding map $T$ with the limit set of any Fuchsian group (see [BS]), and the quasiconformal deformation induces an expanding map on $\Lambda$.

We show that the Hausdorff dimension of the limit sets $\Lambda$ of both Schottky and quasifuchsian groups can be efficiently calculated via a knowledge of the derivatives $DT^n(z)$, evaluated at periodic points $T^n z = z$.

**Theorem 1.** (Kleinian groups) *Let $\Gamma$ be a finitely generated non-elementary convex co-compact Schottky or quasifuchsian group, with associated limit set $\Lambda$. Let $T : \Lambda \to \Lambda$ be the associated dynamical system. For each $N \geq 1$ we can explicitly define a function $\Delta_N$, using only the derivatives $DT^n(z)$ evaluated at period-$n$ points $z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if $s_N$ is the largest real zero of $\Delta_N$ then*

$$|dim(\Lambda) - s_N| \leq C\delta^{N^{3/2}}.$$

*Remark.* The convex cocompact assumption in Theorem 1 corresponds to $\Gamma$ not containing parabolic elements.

The functions $\Delta_N$ are defined as follows. First let

$$a_n = \frac{1}{n} \sum_{T^n z = z} \frac{|DT^n(z)|^{-s}}{\det\left(I - [DT^n(z)]^{-1}\right)},$$

where the summation is over all period-$n$ points $z$, $DT^n(z)$ denotes the derivative of $T^n$ at such a point, and $|DT^n(z)|$ denotes the modulus of the real (as opposed to isometry) part of this derivative. Then define $\Delta_N$ by

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1, \ldots, n_m) \\ n_1 + \cdots + n_m = n}} \frac{(-1)^m}{m!} a_{n_1} \ldots a_{n_m},$$

where the second summation is over all ordered $m$-tuples of positive integers whose sum is $n$.

*Example 1.* The following family of reflection groups (see §6 for more details) was considered by McMullen [McM3]. Consider three circles $C_0, C_1, C_2 \subset \mathbb{C}$ of equal radius, arranged symmetrically around $S^1$, each intersecting $S^1$ orthogonally, and meeting $S^1$ in an arc of length $\theta$, where $0 < \theta < 2\pi/3$ (see Figure 2). Let $\Lambda_\theta \subset \mathbb{S}^1$ denote the limit set associated

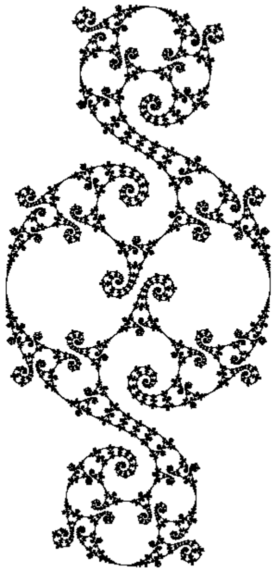

FIGURE 1 . Limit set for a quasifuchsian group

to the group $\Gamma_\theta$ of transformations given by reflection in these circles. For example, with the value $\theta = \pi/6$ we show that the dimension of the limit set $\Lambda_{\pi/6}$ is

$$\dim(\Lambda_{\pi/6}) = 0.18398306124833918694118127344474173288\ldots$$

which is empirically accurate to the 38 decimal places given.

Consider next the Julia set $\mathcal{J}$ of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We shall be interested in those Julia sets $\mathcal{J}$ which are hyperbolic (i.e. $\inf_{z\in\mathcal{J}} |(f^n)'(z)| > 1$ for some $n$). The periodic point algorithm allows the accurate computation of Hausdorff dimension of such Julia sets, or more generally the Julia sets of any hyperbolic holomorphic Markov map (i.e. one whose symbolic dynamics is described by a subshift of finite type; see §1 for the definition). A feature of the approach is that we exploit the *real-analyticity* of the map $f$. That is, we will identify $\mathbb{C}$ with $\mathbb{R}^2$, and think of $\mathcal{J}$ as lying in the real section $\mathbb{R}^2$ of $\mathbb{C}^2$, with $T(x, y) = (\mathrm{Re}\,(f(x + iy)), \mathrm{Im}\,(f(x + iy)))$ an analytic map on some suitable domain in $\mathbb{C}^2$.

**Theorem 2.** (Julia sets) *Let $\mathcal{J}$ be the Julia set of a hyperbolic holomorphic Markov map $f : \mathcal{J} \to \mathcal{J}$. For each $N \geq 1$ we can explicitly define a function $\Delta_N$, using only the derivatives $(f^n)'(z)$ evaluated at period-$n$ points $z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if $s_N$ is the largest real zero of $\Delta_N$ then*

$$|dim(\mathcal{J}) - s_N| \leq C\delta^{N^{3/2}}.$$

This method is effective in computing the dimension of many hyperbolic Julia sets. The Julia sets $\mathcal{J}_c$ of quadratic polynomials $f_c(z) = z^2 + c$ are of particular interest.

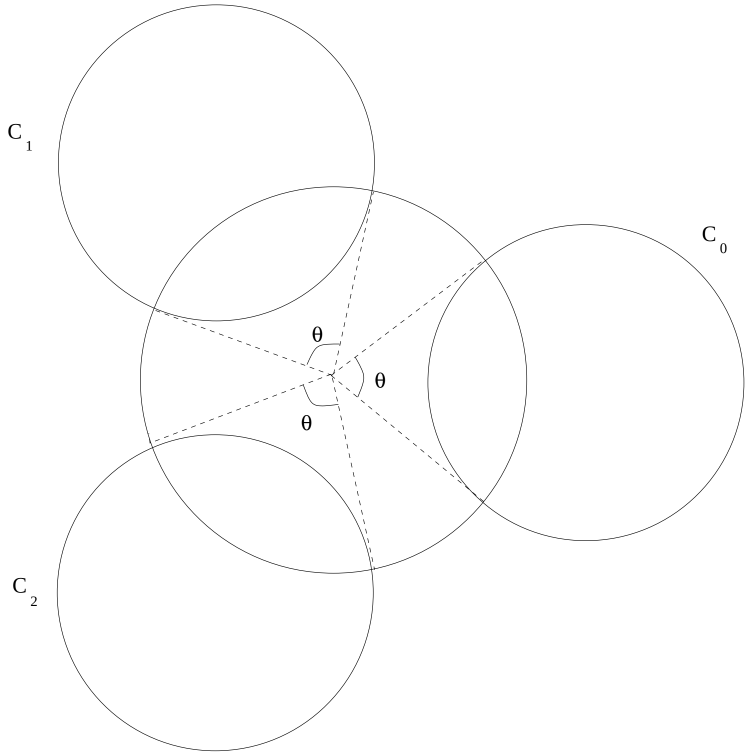


FIGURE 2. Reflection in three circles

*Example 2.* For the quadratic map $f_{i/4}(z) = z^2 + \frac{i}{4}$, the Hausdorff dimension of the associated Julia set $\mathcal{J}_{i/4}$ is given by

$$\dim(\mathcal{J}_{i/4}) \approx 1.0231992890309691251\ldots,$$

where the first 16 decimal digits are empirically correct.

More generally, our approach gives super-exponential convergence to the Hausdorff dimension of any compact set $X$ which lies in a $C^\omega$ $d$-manifold and is invariant under a dynamical system $T : X \to X$ satisfying

(1) *Markov dynamics:* On a symbolic level the dynamics is faithfully coded by a subshift of finite type;
(2) *Hyperbolicity:* There exists some $C > 1$ such that $|T'(x)| > C$ for all $x \in X$;
(3) *Conformality:* $T$ is a conformal map;
(4) *Analyticity:* $T$ is real-analytic;
(5) *Local maximality:* For any sufficiently small open neighbourhood $U$ of the invariant set $X$ we have $X = \cap_{n=0}^{\infty} T^{-n}U$ (such an $X$ is sometimes called a *repeller*).

Note that in dimension two or higher, property (4) is implied by (3).

**Theorem 3.** (General Case) *Let $X \subset M$ be a locally maximal compact invariant set for a conformal real-analytic hyperbolic Markov map $T : X \to X$, where $M$ is a $C^\omega$ manifold of dimension $d \in \mathbb{N}$. For each $N \geq 1$ we can explicitly define a function $\Delta_N$, using only the*

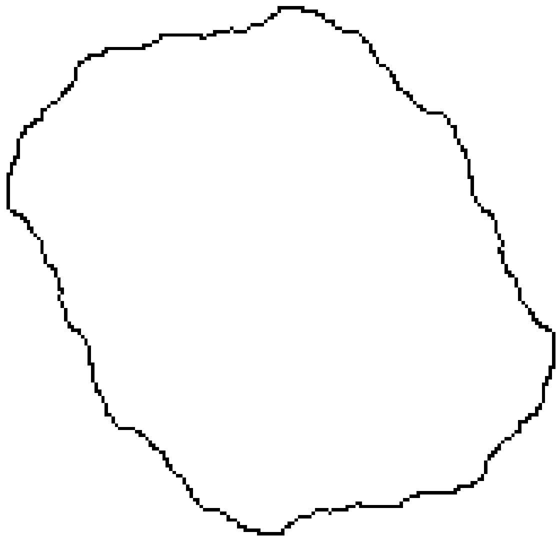

FIGURE 3. The Julia set for $f_{i/4}(z) = z^2 + \frac{i}{4}$

*derivatives $DT^n(z)$ evaluated at period-n points $z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if $s_N$ is the largest real zero of $\Delta_N$ then*

$$|dim(X) - s_N| \leq C\delta^{N^{1+1/d}}.$$

In view of this, we see that stronger results than Theorems 1 and 2 can be proved in the case where the limit set actually lies in a one (real) dimensional submanifold.

**Corollary 3.1.** (Faster convergence on submanifolds)

(1) *Let $\Lambda$ be the limit set of a classical Fuchsian Schottky group, with associated dynamical system $T : \Lambda \to \Lambda$. For each $N \geq 1$ we can explicitly define a function $\Delta_N$, using only the derivatives $DT^n(z)$ evaluated at period-n points $z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if $s_N$ is the largest real zero of $\Delta_N$ then*

$$|dim(\Lambda) - s_N| \leq C\delta^{N^2}.$$

(2) *Let $f_c : \mathcal{J}_c \to \mathcal{J}_c$ be the quadratic map $f_c(z) = z^2 + c$ restricted to the Julia set $\mathcal{J}_c$, where the real parameter $c < -2$. For each $N \geq 1$ we can explicitly define a function $\Delta_N$, using only the derivatives $(f_c^n)'(z)$ evaluated at period-n points $z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if $s_N$ is the largest real zero of $\Delta_N$ then*

$$|dim(\mathcal{J}_c) - s_N| \leq C\delta^{N^2}.$$

Returning to Example 1, we see that since $\Lambda_\theta \subset S^1$ then part (1) of Corollary 3.1 means $\dim(\Lambda_\theta)$ can be approximated at rate $O(\delta^{N^2})$, for some $0 < \delta < 1$.

In the quadratic family $f_c$, for real parameters $c < -2$ the Julia set $\mathcal{J}_c$ is hyperbolic and also lies on the real line. By part (2) of Corollary 3.1 we have the improved convergence rate $O(\delta^{N^2})$. For example taking $c = -5$ the periodic point algorithm yields

$$\dim(\mathcal{J}_{-5}) = 0.48479829443816043053839847...$$

where these 26 decimal places are (empirically) accurate.

All the dynamical systems we consider are *conformal*. Since every *one-dimensional* expanding map is conformal, the periodic point algorithm applies to a wide class of fractal sets lying in the real line. Here, however, we need the extra real-analyticity hypothesis (a property which is *automatically* satisfied by higher-dimensional conformal maps).

*Analytic cookie-cutters.* Many fractals in the line arise as limit sets of so-called cookie-cutters. These are special cases of the iterated function schemes (defined in §1) we consider, and are treated in [Fa2, Ch. 4]. Examples of particular interest, whose Hausdorff dimension can all be calculated via the periodic point algorithm, are the following:

(i) Bounded continued fractions. The classic example is the set $E_2$ considered in [JP1]. More generally, the set of reals whose continued fraction expansions have partial quotients constrained to lie in some finite alphabet is a Cantor set.
(ii) Limit sets of Hecke groups (see [McM3]).
(iii) Julia sets for Blaschke products, for example the family $f_t(z) = z/t - 1/z$ considered in [McM3].

Theorems 1-3 are special cases of Theorem 4, which appears in §3.6, and is formulated in terms of conformal iterated function schemes. These iterated function schemes are introduced in §1. In §2 we review the definition of Hausdorff dimension and Bowen's pressure formula. §3 is the key section, in which we introduce tools such as the transfer operator, Fredholm determinants, and trace formulae, and prove our main results. In §4 we apply these results to Schottky and quasifuchsian groups, and in §5 to Julia sets of holomorphic maps. In §6 we treat in detail the reflection group family of Example 1, and show in §7 how the periodic point algorithm yields rigorous dimension estimates. In §8 we use the algorithm to obtain numerical estimates on the dimension of Julia sets for quadratic maps. In §9 we survey some other examples to which our algorithm can be applied. In Appendix A we discuss the practical implementation of the algorithm (see also the website [JP2]). In Appendix B we prove a lemma about contraction ratios of iterated function schemes, useful for estimating the constant $\delta$ in the theorems.

## 1. Conformal iterated function schemes

In this section we shall introduce notation in order to formulate results in their most general context. We will then specialise to particular examples in later sections.

The defining property of any conformal map $\psi : U \subset \mathbb{R}^d \to \mathbb{R}^d$ is that at each point $z \in U$, its derivative $D\psi(z)$ is the product of a positive real number, denoted by $|D\psi(z)|$, and an isometry $M_\psi(z) \in SO(d)$. The map $z \mapsto |D\psi(z)|$ is called the *conformal derivative* of the map $\psi$.

*Definition.* Let $U_1, \ldots, U_k \subset \mathbb{R}^d$ be a finite collection of subsets of $\mathbb{R}^d$, each of which is the closure of its own interior. For $k \geq 2$, let $A$ be a $k \times k$ aperiodic (i.e. some power of $A$ is strictly positive) matrix with entries either 0 or 1. Assume that for every pair of symbols $i, j \in \{1, \ldots, k\}$ such that $A(i,j) = 1$ (in this case we say the pair $(i,j)$ is *admissible*) we have associated a real analytic conformal map $\phi_{ji} : U_i \to U_j$ such that:

(i) The closure of the image $\phi_{ji}(U_i)$ lies inside $U_j$ (i.e. $\phi_{ji}(U_i) \subset \mathrm{int}(U_j)$) and
(ii) the map $\phi_{ji}$ is a strict contraction (i.e. there exists $0 < \theta < 1$ with $|D\phi_{ji}(z)| \leq \theta$ for all $z \in U_i$).

The collection of maps $\{\phi_{ji} : A(i,j) = 1\}$ is called a *(real-analytic conformal) iterated function scheme* on $\mathbb{R}^d$.

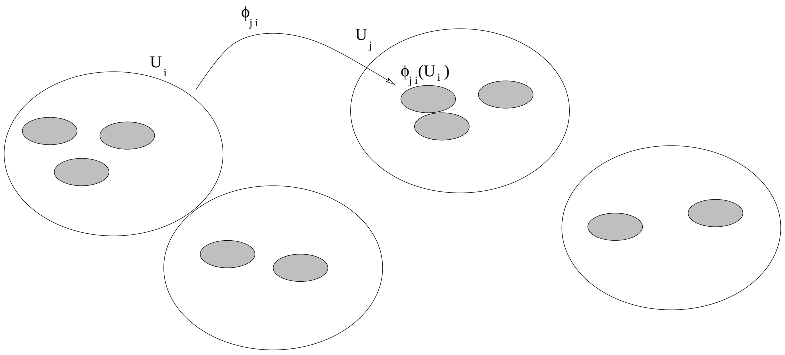


FIGURE 4 . Regions and contractions

Now suppose $M$ is a real-analytic manifold of dimension d. By analogy with the above, suppose we have subsets $U_1, \ldots, U_k \subset M$, each of which is the closure of its own interior, a $k \times k$ aperiodic 0-1 matrix $A$, and a collection of $C^\omega$ conformal maps $\phi_{ji} : U_i \to U_j$ corresponding to admissible pairs $(i,j)$, and satisfying (i), (ii) above. We call this a *(real-analytic conformal) iterated function scheme* on $M$. To say the $\phi_{ji}$ are $C^\omega$ and conformal means precisely that there is a $C^\omega$ structure on $M$ consisting of a a finite collection of connected open sets $V_\alpha \subset M$ and $C^\omega$ diffeomorphisms $f_\alpha : V_\alpha \to \mathbb{R}^d$ such that, if the $U_i$ are sufficiently small so that each $U_i$ is contained in some $V_{\alpha_i}$ (we may always assume this is true, by replacing the collection $\{U_i\}$ by some suitable refinement) then each $\psi_{ji} := f_{\alpha_j} \circ \phi_{ji} \circ f_{\alpha_i}^{-1}$ is a conformal $C^\omega$ map from $W_i := f_{\alpha_i} U_i \subset \mathbb{R}^d$ to $W_j := f_{\alpha_j} U_j \subset \mathbb{R}^d$. Each $\psi_{ji}$ is moreover a strict contraction, so the collection of sets $W_i$ and maps $\psi_{ji}$ form a conformal iterated function scheme on subsets of $\mathbb{R}^d$, whose limit set has the same Hausdorff dimension as the original limit set in $M$ (since in particular the $f_\alpha$ are bi-Lipschitz). In this way we can, and will, always make the simplifying assumption that our iterated function scheme is defined in $\mathbb{R}^d$.

An ordered $(n+1)$-tuple of symbols $\underline{i} = (i_1, \ldots, i_{n+1}) \in \{1, \ldots, k\}^{n+1}$ such that $A(i_j, i_{j+1}) = 1$ for $j = 1, \ldots, n$ is called an *admissible string*. For such a string $\underline{i}$ we let

$$\phi_{\underline{i}} = \phi_{i_{n+1} i_n} \circ \ldots \circ \phi_{i_3 i_2} \circ \phi_{i_2 i_1}$$

denote the $n$-fold composition of the corresponding contractions. We write $|\underline{i}| = n + 1$.

*Definition.* The *limit set* $\Lambda \subset \cup_{i=1}^k U_i$ of an iterated function scheme is defined by

$$\Lambda = \bigcap_{n=1}^{\infty} \overline{\bigcup_{|\underline{i}|=n+1} \phi_{\underline{i}}(U_{i_1})}.$$

If $i_1 = j = i_{n+1}$ then the contraction $\phi_{\underline{i}} : U_j \to U_j$ has a unique fixed point, which we denote by $z_{\underline{i}}$. Let $\mathrm{Fix}(n) = \{\underline{i} : |\underline{i}| = n+1, i_1 = i_{n+1}\}$ denote the set of length-$(n+1)$ strings of this form, and $\mathcal{F}ix(n) = \{z_{\underline{i}} : \underline{i} \in \mathrm{Fix}(n)\}$ the set of corresponding fixed points.

The above notion of an iterated function scheme is somewhat more general than the usual definition (see for example [MU1]). More commonly, the contractions $\phi_{ji} : U_i \to U_j$ are assumed to only depend on the target set $U_j$. This is the case, for example, if the contractions arise as inverse branches of an expanding map, or as cookie-cutter systems. Since this more common scenario is our primary motivation, corresponding to the Julia set and Kleinian limit set cases dealt with later, we now briefly indicate the connection between the two set-ups.

Let $T : X \to X$ be a conformal real analytic expanding map where $X \subset \mathbb{R}^d$ is a closed, $T$-invariant, locally maximal hyperbolic subset. We can choose a *Markov partition*, consisting of a collection of sets $X_1, \ldots, X_k \subset X$ covering $X$ such that the restriction of $T$ to each $X_j$ is injective, and the image $T(X_j)$ is a union of the form $X_{i_1} \cup \ldots \cup X_{i_r}$. We also require that the sets $X_1, \ldots X_k$ have pairwise disjoint interiors, and that each $X_i$ is the closure of its own interior. We can naturally extend $T$ to an analytic expanding map $\hat{T}$ on the disjoint union $\cup_i X_i$. The map $\hat{T}$ is called a *Markov map*. We then define a $k \times k$ transition matrix $A$, where $A(i,j) = 1$ if $\hat{T} X_j \cap X_i \neq \emptyset$, and $A(i,j) = 0$ otherwise. If $A(i,j) = 1$ then we define a contracting map $\phi_{ji} : X_i \to X_j$ by requiring that $\hat{T} \circ \phi_{ji} = id|_{X_i}$. The collection $\{\phi_{ji} : A(i,j) = 1\}$ is then an iterated function scheme.

In fact, if $\hat{T} X_j = X_{i_1} \cup \ldots \cup X_{i_r}$, then it is common to define an inverse branch $S_j : X_{i_1} \cup \ldots \cup X_{i_r} \to X_j$ by $T \circ S_j = id|_{X_{i_1} \cup \ldots \cup X_{i_r}}$ (so that $\phi_{ji} = S_j|_{X_i}$ for each admissible $(i,j)$). For an admissible $n$-tuple $\underline{j} = (j_1, \ldots, j_n)$ we will also define $S_{\underline{j}} = S_{j_n} \circ \cdots \circ S_{j_1} : X_{i_1} \cup \ldots \cup X_{i_r} \to X_{j_n}$, and note that $\hat{T} \circ S_{\underline{j}} = id|_{X_{i_1} \cup \ldots \cup X_{i_r}}$.

Note that such a contraction $S_{\underline{j}}$ has a unique fixed point. This fixed point is precisely $z_{\underline{i}}$ (i.e. the fixed point for $\phi_{\underline{i}}$), where $\underline{i} = (j_n, j_1, \ldots, j_n)$. Thus there is a one-to-one correspondence between the set $\mathrm{Fix}(n)$ and the set of *all* admissible length-$n$ words $\underline{j}$. Moreover, the set $\mathcal{F}ix(n)$ of fixed points $z_{\underline{i}}$ of length-$n$ compositions of contractions is precisely the set of period-$n$ points for the map $\hat{T}$. Throughout we adopt the convention of counting periodic points with multiplicities corresponding to $\hat{T}$, rather than $T$. The precise connection between the periodic points for $T$ and $\hat{T}$ is well understood, using Markov partitions and Manning's combinatorial lemma [Ma].

## 2. Hausdorff dimension and Bowen's pressure formula

Let us recall the definition of Hausdorff dimension.

*Definition.* For $X \subset \mathbb{R}^d$, and $s \geq 0$ we let $H_\varepsilon^s(X) = \inf_{\mathcal{U}} \{\sum_i \mathrm{diam}(U_i)^s\}$ where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ such that $\mathrm{diam}(U_i) \leq \varepsilon$. We define the

*$s$-dimensional Hausdorff measure* of $X$ by $H^s(X) = \lim_{\varepsilon\to 0} H^s_\varepsilon(X)$ and then the *Hausdorff dimension* of $X$ is defined by

$$\dim(X) = \inf\{s : H^s(X) = 0\}.$$

The sets we consider will always be compact, and invariant under some dynamical system. There are various other notions of dimension (see [Fa1]). For the conformal iterated function schemes we consider, the Hausdorff dimension is equal to the box dimension, for example. For dynamically defined sets, the following definition is useful.

*Definition.* Given a conformal iterated function scheme, we define its *pressure function* $p : \mathbb{R} \to \mathbb{R}$ by

$$p(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\underline{i}\in \mathrm{Fix}(n)} |D\phi_{\underline{i}}(z_{\underline{i}})|^s.$$

If our iterated function scheme is derived from an expanding Markov map $T$, in the way described in §1, then $p(s)$ is just the usual pressure (see [Wal])

$$P(-s\log|\hat{T}'|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\hat{T}^n z = z} |D\hat{T}^n(z)|^{-s}$$

of the function $s \mapsto -s\log|\hat{T}'|$ with respect to the Markov dynamical system $\hat{T}$.

**Proposition 1.** *Let $\Lambda$ be the limit set of a conformal iterated function scheme. Then the Hausdorff dimension $\dim(\Lambda)$ is the unique zero of the strictly decreasing real function $s \mapsto p(s)$.*

A proof of the above result appears in the article of Bowen [Bo], in the special case of quasifuchsian groups. However, as Ruelle observed in [Ru3] for Julia sets, the proof applies in greater generality (see also [Pe], [PU]).

## 3. Proof of Main Theorem

A key tool in our analysis is a family of bounded linear transfer operators $\mathcal{L}_s$ on a Banach space of bounded holomorphic functions. The transfer operator is defined in terms of our iterated function scheme. A key property of $\mathcal{L}_s$ is that it is a nuclear operator (Proposition 2), so that we can define its Fredholm determinant $\det(I - z\mathcal{L}_s)$, an entire function of $z$ (Proposition 4). The fact that $\mathcal{L}_s$ is defined as a sum of composition operators means that the traces $\mathrm{tr}(\mathcal{L}_s^n)$ can be evaluated in terms of fixed points of our iterated function scheme (Proposition 3), and these traces are then used in §3.6 to give approximations to $\det(I - z\mathcal{L}_s)$.

### 3.1 Complexification of the iterated function scheme.

We have the natural identification

$$\mathbb{R}^d = \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times i\mathbb{R}^d = \mathbb{C}^d.$$

For each symbol $i \in \{1, \ldots, k\}$, let us choose some open polydisc $D_i = D_i^{(1)} \times \ldots \times D_i^{(d)} \subset \mathbb{C}^d$, where each $D_i^{(l)}$ is an open disc in $\mathbb{C}$, such that $U_i \times \{0\} \subset D_i$.

All our maps are real-analytic and contracting. Therefore by considering a refined Markov partition if necessary, we may assume that for each admissible pair $(i, j)$, the maps $\phi_{ji}$, $|D\phi_{ji}(\cdot)|$ all extend holomorphically to maps $D_i \to D_j$, such that both

$$\overline{\phi_{ji}(D_i)} \subset D_j,$$

and

$$\sup_{z \in D_i} |D\phi_{ji}(z)| < 1. \tag{3.1}$$

By standard abuse of notation we let $\phi_{ji}$, $|D\phi_{ji}(\cdot)|$ denote these holomorphic extensions to $D_i$. Similarly, we define the maps $\phi_{\underline{i}}$, $|D\phi_{\underline{i}}(\cdot)|$ to be the holomorphic extensions of the corresponding previously defined restrictions to the real section.

Note that if $\underline{i} \in \mathrm{Fix}(n)$ (i.e. if $i_1 = i_{n+1}$) then the contraction $\phi_{\underline{i}}$ has $z_{\underline{i}}$ as a unique fixed point. In particular $z_{\underline{i}}$ lies in the real section $\mathbb{R}^d$.

Let $D = \coprod_{i=1}^k D_i$ denote the disjoint union of our polydiscs.

Now to each polydisc $D_i = D_i^{(1)} \times \ldots \times D_i^{(d)} \subset \mathbb{C}^d$ in our disjoint union $D$, we want to associate a concentric open polydisc $E_i = E_i^{(1)} \times \ldots \times E_i^{(d)}$, where each $E_i^{(l)}$ has radius strictly larger than $D_i^{(l)}$.

By (3.1) we know that for each admisible pair $(i, j)$, $|D\phi_{ji}(\cdot)|$ extends holomorphically to some open neighbourhood $D_{i,j}$ of $D_i$, with the property that

$$|D\phi_{j,i}(\cdot)| < 1 \text{ on } D_{i,j}. \tag{3.2}$$

Moreover, since $\phi_{ji}$ is a contraction then $\phi_{ji}^{-1} D_j$ contains some open neighbourhood of $D_i$.

So defining $E_i$ to be the largest polydisc, concentric with $D_i$, and contained in

$$\bigcap_{j: A(i,j)=1} \phi_{ji}^{-1} D_j \cap D_{i,j},$$

ensures it has the property that each of its component discs has radius strictly larger than the corresponding component disc in $D_i$.

Moreover, we have that $\phi_{ji}$ extends holomorphically to $E_i$, and the important property that

$$\overline{\phi_{ji} E_i} \subset D_i \text{ for each admissible pair } (i, j). \tag{3.3}$$

Let $E = \coprod_{i=1}^k E_i$ denote the disjoint union of the larger polydiscs.

**3.2 Transfer operators.**

For any open set $U$, let $\mathcal{A}_\infty(U)$ denote the Banach space of those holomorphic functions on $U$ which are bounded on the closure $\overline{U}$, equipped with the supremum norm. We will consider such spaces for the various choices $U = D_i, E_i, D, E$.

For any $s \in \mathbb{C}$, and any admissible pair $(i, j)$, define the *weight function* $w_{s,(j,i)} \in \mathcal{A}_\infty(E_i)$ by

$$w_{s,(j,i)}(z) = |D\phi_{ji}(z)|^s. \tag{3.4}$$

Then define the bounded linear operator $\mathcal{L}_{s,(j,i)} : \mathcal{A}_\infty(D_j) \to \mathcal{A}_\infty(D_i)$ by

$$\mathcal{L}_{s,(j,i)} g(z) = g(\phi_{ji} z) w_{s,(j,i)}(z). \tag{3.5}$$

Since $w_{s,(j,i)} \in \mathcal{A}_\infty(E_i)$ then by virtue of the inclusion (3.3) we have the inclusion

$$\mathcal{L}_{s,(j,i)} \left(\mathcal{A}_\infty(D_j)\right) \subset \mathcal{A}_\infty(E_i). \tag{3.6}$$

For a fixed $i$ we sum over all (admissible) composition-type operators $\mathcal{L}_{s,(j,i)}$ to form the *component transfer operator* $\mathcal{L}_{s,i}$. That is, we define

$$\mathcal{L}_{s,i} h(z) = \sum_{j:A(i,j)=1} h(\phi_{ji} z) w_{s,(j,i)}(z). \tag{3.7}$$

Note that $\mathcal{L}_{s,i}$ is naturally an operator $\mathcal{A}_\infty(\coprod_{j:A(i,j)=1} D_j) \to \mathcal{A}_\infty(D_i)$, and clearly also defines an operator $\mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D_i)$. Indeed, since (3.6) holds for each admissible $(i,j)$ then

$$\mathcal{L}_{s,i} \left(\mathcal{A}_\infty(D)\right) \subset \mathcal{A}_\infty(E_i). \tag{3.8}$$

Now we define the *transfer operator* $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ by setting

$$\mathcal{L}_s h|_{D_i} = \mathcal{L}_{s,i} h$$

for each $h \in \mathcal{A}_\infty(D)$ and each $i \in \{1, \ldots, k\}$.

The inclusion (3.8) means that

$$\mathcal{L}_s \left(\mathcal{A}_\infty(D)\right) \subset \mathcal{A}_\infty(E). \tag{3.9}$$

Note that if we define $\mathcal{M}_{s,i} : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ by

$$\mathcal{M}_{s,i} h|_{D_i} = \mathcal{L}_{s,i} h, \quad \mathcal{M}_{s,i} h|_{D_j} = 0 \text{ if } j \in \{1, \ldots, k\} \setminus \{i\}, \tag{3.10}$$

then we can write

$$\mathcal{L}_s = \sum_{i=1}^{k} \mathcal{M}_{s,i}. \tag{3.11}$$

The following result is standard (see [Ru2]).

**Lemma 1.** *For real $s$, the transfer operator $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ has spectral radius $e^{p(s)}$. The value $e^{p(s)}$ is the unique eigenvalue of maximum modulus, and is simple and isolated.*

*Remark.* As is well known, the spectral properties of the transfer operator $\mathcal{L}_s$ depend strongly on the underlying Banach space. When acting on the space $C^\alpha(\Lambda)$ of $\alpha$-Hölder functions, $\mathcal{L}_s$ again has spectral radius $e^{p(s)}$, the unique eigenvalue of maximum modulus, but is not a compact operator. However the operator does have a spectral gap; that is, its essential spectral radius is strictly smaller than $e^{p(s)}$ (see [Ru2]). In §3.3 it will be shown that when acting on $\mathcal{A}_\infty(D)$, the operator $\mathcal{L}_s$ is *nuclear*, which in particular implies its compactness.

**3.3 Nuclear operators.**

*Definition.* A linear operator $L : B \to B$ on a Banach space $B$ is called *nuclear* if there exist $u_n \in B$, $l_n \in B^*$ (with $||u_n|| = 1$ and $||l_n|| = 1$) and $\sum_{n=0}^{\infty} |\rho_n| < +\infty$ such that

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n, \qquad \text{for all } v \in B. \tag{3.12}$$

$L$ is said to be *nuclear of order $p$* if $\sum_n |\rho_n|^p < \infty$, and *nuclear of order zero* if this holds for all $p > 0$.

In this section we will show that the transfer operator $\mathcal{L}_s$ is nuclear of order zero. First we need some notation. Let us fix $i \in \{1, \dots, k\}$. Let $z^{(i)} = (z_1^{(i)}, \dots, z_d^{(i)})$ denote the common centre of the two polydiscs. Let $\sigma_i^{(l)}$ be the radius of the $l^{th}$ component disc $D_i^{(l)}$.

Let $\Gamma_i = \Gamma_i^{(1)} \times \dots \times \Gamma_i^{(d)}$ be a polycircle, centred at $z^{(i)}$, exterior to $D_i$, yet interior to $E_i$, such that the radii $\tau_i^{(l)}$ of the circles $\Gamma_i^{(l)}$ satisfy $\gamma_i^{(l)} := \sigma_i^{(l)}/\tau_i^{(l)} < 1$ (possible by choosing $\Gamma_i^{(l)}$ sufficiently close to the boundary of $E_i^{(l)}$.

We call $\gamma_i^{(l)}$ the *contraction ratio* in the $l^{th}$ component.

**Proposition 2.** *The transfer operator $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is nuclear of order zero.*

*Poof.* Consider the component transfer operator $\mathcal{L}_{s,i} : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D_i)$.

For $h \in \mathcal{A}_\infty(D)$ we know by (3.8) that not only is $\mathcal{L}_{s,i}h$ holomorphic on the polydisc $D_i = D_i^{(1)} \times \dots \times D_i^{(d)}$, but it is actually holomorphic on the larger concentric polydisc $E_i = E_i^{(1)} \times \dots \times E_i^{(d)}$.

Therefore by the multi-dimensional Cauchy formula we can integrate around the polycircle $\Gamma_i$ to obtain the representation

$$\mathcal{L}_{s,i}h(z) = \mathcal{L}_{s,i}h(z_1, \dots, z_d) = \frac{1}{(2\pi\sqrt{-1})^d} \int_{\Gamma_i} \frac{\mathcal{L}_{s,i}h(\xi_1, \dots, \xi_d)}{(\xi_1 - z_1)\dots(\xi_d - z_d)} d\xi_1 \dots d\xi_d.$$

Expanding each

$$(\xi_l - z_l)^{-1} = (\xi_l - z_l^{(i)})^{-1} \left(1 - \frac{z_l - z_l^{(i)}}{\xi_l - z_l^{(i)}}\right)^{-1} = (\xi_l - z_l^{(i)})^{-1} \sum_{r_l=0}^{\infty} \left(\frac{z_l - z_l^{(i)}}{\xi_l - z_l^{(i)}}\right)^{r_l}$$

then gives us

$$\mathcal{L}_{s,i}h(z) = \sum_{r_1=0}^{\infty} \dots \sum_{r_d=0}^{\infty} m_{(r_1,\dots,r_d)}^{(i)}(h)\, v_{(r_1,\dots,r_d)}^{(i)}(z_1, \dots, z_d). \tag{3.13}$$

Here the linear functionals $m_{(r_1,\dots,r_d)}^{(i)} \in \mathcal{A}_\infty(D)^*$ are given by

$$m_{(r_1,\dots,r_d)}^{(i)}(h) = \frac{1}{(2\pi\sqrt{-1})^d} \int_{\Gamma_i} \frac{\mathcal{L}_{s,i}h(\xi_1, \dots, \xi_d)}{(\xi_1 - z_1^{(i)})^{r_1+1} \dots (\xi_d - z_d^{(i)})^{r_d+1}} d\xi_1 \dots d\xi_d,$$

and the functions $v^{(i)}_{(r_1,\dots,r_d)} \in \mathcal{A}_\infty(D_i)$ are monomials

$$v^{(i)}_{(r_1,\dots,r_d)}(z) = v^{(i)}_{(r_1,\dots,r_d)}(z_1,\dots,z_d) = (z_1 - z^{(i)}_1)^{r_1} \dots (z_d - z^{(i)}_d)^{r_d}.$$

For each $v^{(i)}_{(r_1,\dots,r_d)} \in \mathcal{A}_\infty(D_i)$ we can naturally associate a function $v_{(r_1,\dots,r_d),i} \in \mathcal{A}_\infty(D)$ defined by

$$v_{(r_1,\dots,r_d),i}|_{D_i} = v^{(i)}_{(r_1,\dots,r_d)}, \quad v_{(r_1,\dots,r_d),i}|_{D_j} = 0 \text{ for } j \in \{1,\dots,k\} \setminus \{i\}.$$

This then allows us to combine all the component transfer operators $\mathcal{L}_{s,i}$, and express the transfer operator $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ as

$$\mathcal{L}_s h(z) = \sum_{i=1}^{k} \sum_{r_1=0}^{\infty} \dots \sum_{r_d=0}^{\infty} m^{(i)}_{(r_1,\dots,r_d)}(h)\, v_{(r_1,\dots,r_d),i}(z) \tag{3.14}$$

for $h \in \mathcal{A}_\infty(D)$, $z \in D$.

Let us normalise the functions and functionals in (3.14) by setting

$$l^{(i)}_{(r_1,\dots,r_d)} = m^{(i)}_{(r_1,\dots,r_d)}/||m^{(i)}_{(r_1,\dots,r_d)}||_\infty, \quad u_{(r_1,\dots,r_d),i} = v_{(r_1,\dots,r_d),i}/||v_{(r_1,\dots,r_d),i}||_\infty.$$

We then set $\lambda_{(r_1,\dots,r_d),i} = ||m^{(i)}_{(r_1,\dots,r_d)}||_\infty ||v_{(r_1,\dots,r_d),i}||_\infty$, so that (3.14) becomes

$$\mathcal{L}_s h = \sum_{i=1}^{k} \sum_{r_1=0}^{\infty} \dots \sum_{r_d=0}^{\infty} \lambda_{(r_1,\dots,r_d),i}\, l^{(i)}_{(r_1,\dots,r_d)}(h)\, u_{(r_1,\dots,r_d),i}. \tag{3.15}$$

We have the estimate

$$\begin{aligned}\lambda_{(r_1,\dots,r_d),i} &\le \frac{||\mathcal{L}_{s,i}||_\infty}{(2\pi)^d \tau^{(1)}_i \dots \tau^{(d)}_i} \left(\gamma^{(1)}_i\right)^{r_1} \dots \left(\gamma^{(d)}_i\right)^{r_d} \\ &= A \left(\gamma^{(1)}_i\right)^{r_1} \dots \left(\gamma^{(d)}_i\right)^{r_d},\end{aligned}$$

where $A = A(d,s,i)$ only depends on $d, s, i$, and on the radii of the component discs of $D_i$ and $E_i$.

In particular, setting $\gamma_i = \max\{\gamma^{(1)}_i, \dots, \gamma^{(d)}_i\} < 1$ we have

$$\lambda_{(r_1,\dots,r_d),i} \le K \gamma_i^{r_1+\dots+r_d}. \tag{3.16}$$

(Note at this point that there are certainly more refined ways of choosing $\gamma_i$ if we are interested in optimal estimates).

For each $i \in \{1,\dots,k\}$, let the sequence $\{\lambda_{n,i}\}_{n=0}^\infty$ be a non-increasing rearrangement of the multi-sequence $\{\lambda_{(r_1,\dots,r_d),i}\}_{r_1,\dots,r_d=0}^\infty$, and let the sequence $\{\mu_{n,i}\}_{n=0}^\infty$ be a non-increasing rearrangement of the multi-sequence $\{\gamma_i^{r_1+\dots+r_d}\}_{r_1,\dots,r_d=0}^\infty$. Then (3.16) implies that

$$\lambda_{n,i} \le K \mu_{n,i} \tag{3.17}$$

for all $i \in \{1, \dots, k\}$, $n \geq 0$.

Note that there are $\binom{N+d}{d}$ non-negative integer vectors $(r_1, \dots, r_d) \in \mathbb{N}_0^d$ such that $r_1 + \dots + r_d \leq N$ (see [Fri, p. 506]). Therefore the smallest $n$ with $\mu_n = \gamma_i^{N+1}$ is $n = \binom{N+d}{d} = O(N^d)$, and we have

$$\mu_{n,i} = O\left(\gamma_i^{n^{1/d}}\right) \text{ as } n \to \infty \tag{3.18}$$

Together (3.17) and (3.18) give that

$$\lambda_{n,i} = O\left(\gamma_i^{n^{1/d}}\right) \text{ as } n \to \infty \tag{3.19}$$

Defining $\gamma = \max\{\gamma_1, \dots, \gamma_k\} < 1$ (again note that this estimate is not optimal) gives us, for each $i \in \{1, \dots, k\}$,

$$\lambda_{n,i} = O\left(\gamma^{n^{1/d}}\right) \text{ as } n \to \infty. \tag{3.20}$$

So if the sequence $\{\lambda_n\}_{n=0}^\infty$ is a non-increasing rearrangement of the multi-sequence $\{\lambda_{n,i} : (n,i) \in \mathbb{N}_0 \times \{1, \dots, k\}\}$ then setting $\alpha = \gamma^{1/k} < 1$ gives us

$$\lambda_n = O\left(\alpha^{n^{1/d}}\right) \text{ as } n \to \infty. \tag{3.21}$$

Now each of the above non-increasing rearrangements induces an obvious rearrangement of the corresponding multi-sequences of functions and functionals, with the new sequences just inheriting the same subindices. In particular we arrive at a sequence $\{l_n\}_{n=0}^\infty$ of functionals, and a sequence $\{u_n\}_{n=0}^\infty$ of functions, such that the representation (3.15) becomes $\mathcal{L}_s h = \sum_{n=0}^\infty \lambda_n l_n(h) u_n$. Then the estimate (3.21) ensures that $\sum_{n=0}^\infty |\lambda_n|^p < \infty$ for all $p > 0$, so that the operator $\mathcal{L}_s$ is nuclear of order zero.

**3.4 Trace formulae.**

The fact that the operators $\mathcal{L}_s$ are nuclear of order zero means they have well-defined spectral traces (the sum of all eigenvalues, counted with algebraic multiplicities). The key to our method is the following explicit formula for the traces of the powers $\mathcal{L}_s^n$ in terms of the fixed points of our iterated function scheme.

**Proposition 3.** *If $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is the transfer operator associated to a conformal iterated function scheme then*

$$tr(\mathcal{L}_s^n) = \sum_{\underline{i} \in \mathrm{Fix}(n)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{det(I - D\phi_{\underline{i}}(z_{\underline{i}}))},$$

*where $D\phi_{\underline{i}}$ and $|D\phi_{\underline{i}}(\cdot)|$ are the d-dimensional and conformal derivatives, respectively, of the map $\phi_{\underline{i}}$.*

*Proof.* For each admissible string $\underline{i} = (i_1, \dots, i_{n+1})$ let us first define composition-like operators $\mathcal{L}_{s,\underline{i}} : \mathcal{A}_\infty(D_{i_{n+1}}) \to \mathcal{A}_\infty(D_{i_1})$ by

$$\mathcal{L}_{s,\underline{i}} g(z) = g(\phi_{\underline{i}} z) w_{s,\underline{i}}(z), \tag{3.22}$$

where the weight functions $w_{s,\underline{i}} \in \mathcal{A}_\infty(D_{i_1})$ are given by

$$w_{s,\underline{i}}(z) = |D\phi_{\underline{i}}(z)|^s.$$

For a fixed $i_1 = i$, the $n^{th}$ iterate of the component transfer operator $\mathcal{L}_{s,i}$ (see (3.7)) is given by

$$\mathcal{L}_{s,i}^n = \sum_{\substack{|\underline{i}|=n+1 \\ i_1=i}} \mathcal{L}_{s,\underline{i}},$$

where the summation is over those length-$(n+1)$ admissible strings $\underline{i} = (i_1, \ldots, i_{n+1})$ with $i_1 = i$.

Then note that the $n^{th}$ iterates of the operators $\mathcal{M}_{s,i} : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ (defined by (3.10)) satisfy

$$\mathcal{M}_{s,i}^n u|_{D_i} = \mathcal{L}_{s,i}^n u, \quad \mathcal{M}_{s,i}^n u|_{D_j} = 0 \text{ if } j \in \{1, \ldots, k\} \setminus \{i\},$$

so we can express

$$\mathcal{L}_s^n = \sum_{i=1}^k \mathcal{M}_{s,i}^n. \tag{3.23}$$

The additivity of the trace means we then have

$$\begin{aligned} \operatorname{tr}(\mathcal{L}_s^n) &= \sum_{i=1}^k \operatorname{tr}(\mathcal{M}_{s,i}^n) = \sum_{i=1}^k \operatorname{tr}(\mathcal{L}_{s,i}^n) \\ &= \sum_{i=1}^k \sum_{\substack{|\underline{i}|=n+1 \\ i_1=i}} \operatorname{tr}(\mathcal{L}_{s,\underline{i}}) \\ &= \sum_{|\underline{i}|=n+1} \operatorname{tr}(\mathcal{L}_{s,\underline{i}}) \\ &= \sum_{\underline{i}\in \operatorname{Fix}(n)} \operatorname{tr}(\mathcal{L}_{s,\underline{i}}). \end{aligned} \tag{3.24}$$

The last equality in the above follows because if $i_1 \neq i_{n+1}$ then the domain and target spaces of the operator $\mathcal{L}_{s,\underline{i}} : \mathcal{A}_\infty(D_{i_{n+1}}) \to \mathcal{A}_\infty(D_{i_1})$ are not the same, so it has no eigenvalues.

If $\underline{i} \in \operatorname{Fix}(n)$, however, we have the following trace formula for the operators $\mathcal{L}_{s,\underline{i}}$ in terms of the fixed point $z_{\underline{i}}$ of the composition $\phi_{\underline{i}}$,

$$\operatorname{tr}(\mathcal{L}_{s,\underline{i}}) = \frac{w_{s,\underline{i}}(z_{\underline{i}})}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))} = \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}. \tag{3.25}$$

The above formula (3.25) has its origins in the work of Atiyah & Bott [AB] on the Lefschetz fixed point theorem, and in our context is proved in [May1] (see also [May2]). Note that since $\phi_{\underline{i}} : U_{i_1} \to U_{i_n}$ is a contraction, then the determinant $\det(I - D\phi_{\underline{i}}(\underline{z}_{\underline{i}})) > 0$.

Combining (3.24) and (3.25) completes the proof.

**3.5 Fredholm determinants.**

*Definition.* For $s \in \mathbb{C}$ and $z \in \mathbb{C}$ we define the *Fredholm determinant* $\det(I - z\mathcal{L}_s)$ of the transfer operator $\mathcal{L}_s$ by

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_s^n)\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\underline{i} \in \mathrm{Fix}(n)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}\right). \tag{3.26}$$

**Proposition 4.** *The Fredholm determinant $det(I - z\mathcal{L}_s)$ is an entire function of both $z$ and $s$.*

*Proof.* The proof that $\det(I - z\mathcal{L}_s)$ is an entire function of $z$ follows immediately from the fact that $\mathcal{L}_s$ is nuclear [Gr1]. For entirety as a function of $s$ we note that the operators $\mathcal{L}_s$ depend holomorphically on $s$, and then use [Gr2, p. 346].

**Proposition 5.** *For any $s \in \mathbb{C}$, let $\lambda_r(s)$, $r = 1, 2, \ldots$ be an enumeration of the non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities. Then*

$$\det(I - z\mathcal{L}_s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)).$$

*In particular, the set of zeros $z$ of the Fredholm determinant $det(I - z\mathcal{L}_s)$, counted with algebraic multiplicities, is equal to the set of reciprocals of non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities.*

*Proof.* This is classical Fredholm theory, valid for any nuclear operator of order 2/3 [Gr2, p. 355], and in particular for the operators $\mathcal{L}_s$, which are nuclear of order zero.

**Proposition 6.** *Given an iterated function scheme, the Hausdorff dimension $dim(\Lambda)$ of its limit set $\Lambda$ is the largest real zero of the function $s \mapsto det(I - \mathcal{L}_s)$.*

*Proof.* If $s$ is real then by Proposition 1 and Lemma 1, the operator $\mathcal{L}_s$ has simple maximal eigenvalue $e^{p(s)}$, which equals 1 if and only if $s = \dim(\Lambda)$. But Proposition 5 means that 1 is an eigenvalue of $\mathcal{L}_s$ if and only if $s$ is a zero of $\det(I - \mathcal{L}_s)$.

To see that $\dim(\Lambda)$ is the *largest* real zero of $\det(I - \mathcal{L}_s)$, note that the spectral radius of $\mathcal{L}_s$ equals $e^{p(s)}$, so is a decreasing function of $s$. In particular, if $s > \dim(\Lambda)$ then the spectral radius of $\mathcal{L}_s$ is less than 1, so that 1 cannot be an eigenvalue of $\mathcal{L}_s$, and hence cannot be a zero of $\det(I - \mathcal{L}_s)$.

By Proposition 4 we can expand $\det(I - z\mathcal{L}_s)$ as a power series in $z$. The power series coefficients are described by the following lemma, due to Grothendieck [Gr2].

**Lemma 2.** *If $\mathcal{L}_s u(z) = \sum_{n=0}^{\infty} \lambda_n l_n(u) v_n$ is a nuclear representation (cf. (3.12), (3.15)) for the operator $\mathcal{L}_s$, and $det(I - z\mathcal{L}_s) = 1 + \sum_{N=1}^{\infty} d_N(s) z^N$ is the power series expansion of its Fredholm determinant, then*

$$d_N(s) = (-1)^N \sum_{k_1 < \ldots < k_N} \lambda_{k_1} \ldots \lambda_{k_N} \, det\left(l_{i_p}(v_{i_q})\right)_{p,q=1}^{n} \tag{3.27}$$

*where*

$$det\left[l_{i_p}(v_{i_q})\right]_{p,q=1}^{N}=\begin{vmatrix} l_{i_1}(v_{i_1}) & \dots & l_{i_N}(v_{i_1}) \\ \vdots & & \vdots \\ l_{i_N}(v_{i_1}) & \dots & l_{i_N}(v_{i_N}) \end{vmatrix}$$

*denotes the determinant of the $N\times N$ matrix with entries $l_{i_p}(v_{i_q})$, $1\le p,q\le N$, corresponding to functionals $l_{i_p}$ and functions $v_{i_q}$.*

**Proposition 7.** *There exists $0<\delta<1$ such that for all $s\ge 0$ the power series coefficients $d_N(s)$ defined by (3.27) satisfy $d_N(s)=O\left(\delta^{N^{1+1/d}}\right)$ as $N\to\infty$.*

*Proof.* The estimate (3.21) means there exist $0<\alpha<1$, $K>0$ such that $\lambda_n\le K\alpha^{n^{1/d}}$ for all $n\ge 0$. Moreover, by a theorem of Hadamard [Had], any $N\times N$ matrix with entries of modulus at most 1 has determinant with modulus bounded by $N^{N/2}$.

Therefore Lemma 1 gives us

$$|d_N(s)|\le K^N N^{N/2}\sum_{k_1<\dots<k_N}\alpha^{k_1^{1/d}}\dots\alpha^{k_N^{1/d}}. \tag{3.28}$$

Now $\beta_N(\alpha):=\sum_{k_1<\dots<k_N}\alpha^{k_1^{1/d}}\dots\alpha^{k_N^{1/d}}$ is precisely the $N^{th}$ power series coefficient of the infinite product $f_\alpha(z)=\prod_{n=0}^{\infty}(1+\alpha^{n^{1/d}}z)=1+\sum_{N=1}^{\infty}\beta_N(\alpha)z^N$. We would like to show that $\beta_N(\alpha)=O(\delta^{N^{1+1/d}})$ for some $0<\delta<1$, so that the decay of $\beta_N(\alpha)$ dominates the growth of $K^N N^{N/2}$, giving $d_N(s)=O(\delta^{N^{1+1/d}})$. For $d=1$ the term $\beta_N(\alpha)$ can in fact be expressed in closed form, using an $N$-fold geometric summation (see §6.3). For general $d$ the following analysis closely follows that of Fried [Fri, p. 506].

For any $r>0$, Cauchy's estimate gives $\beta_N(\alpha)\le M(r)/r^N$, where $M(r)=M_\alpha(r):=\max_{|z|=r}|f_\alpha(z)|$. Let $m(r)$ denote the number of zeros $z$ of $f_\alpha$ satisfying $|z|\le r$. Since the zeros of $f_\alpha$ are just $z=-\alpha^{-n^{1/d}}$, for $n=0,1,\dots$, we can estimate $m(r)\le a(\log_+ r)^d$, where $a=a_\alpha:=(-\log\alpha)^{-d}>0$. From the infinite product defining $f_\alpha$, it is clear that $f_\alpha$ has genus zero (see [Boas, §2.5, 2.7]). Since also $f(0)=1$, by [Boas, p. 47] we have $\log M(r)\le N(r)+Q(r)$, where $N(r)=\int_0^r\frac{m(t)}{t}dt$ and $Q(r)=r\int_r^\infty\frac{m(t)}{t^2}dt$. Our upper bound on $m(r)$ then gives us

$$\log M(r)\le a\int_1^r\frac{(\log t)^d}{t}dt+ar\int_{\max\{1,r\}}^\infty\frac{(\log t)^d}{t^2}dt.$$

Substituting $u=\log t$, we can evaluate these integrals to give

$$\log M_\alpha(r)\le a_\alpha P(\log r),$$

where $P(x):=\sum_{j=0}^{d+1}\frac{d!}{j!}x^j$. In particular, $P$ is a degree-$(d+1)$ polynomial whose leading coefficient is $1/(d+1)$.

Therefore we have

$$\beta_N(\alpha)\le\frac{M_\alpha(r)}{r^N}\le r^{-N}\exp\left(a_\alpha P(\log r)\right) \tag{3.29}$$

for all $r > 0$. For each $N$ we will evaluate this inequality at $r = r_N = \exp\left((N/a)^{1/d}\right)$. Now $r_N^{-N} = \exp\left(-a^{-1/d}N^{1+1/d}\right)$, while the leading term in $aP(\log r_N) = aP\left((N/a)^{1/d}\right)$ is $\frac{1}{d+1}a^{-1/d}N^{1+1/d}$. Thus

$$r_N^{-N}\exp\left(aP(\log r_N)\right) = \exp\Big((-1+1/(d+1))a^{-1/d}N^{1+1/d} + O(N)\Big) = \varepsilon^{N^{1+1/d}}\exp(O(N)),$$

where $\varepsilon = \exp\left(\frac{-d}{d+1}a^{-1/d}\right) = \alpha^{d/(d+1)} < 1$.

From (3.29) we have $\beta_N(\alpha) \le r_N^{-N}\exp(aP(\log r_N)) = O(\varepsilon_1^{N^{1+1/d}})$ for any $1 > \varepsilon_1 > \varepsilon = \alpha^{d/(d+1)}$, so substituting into (3.28) gives $d_N(s) = O(\delta^{N^{1+1/d}})$ for any $\alpha^{d/(d+1)} < \delta < 1$.

*Remark 1.* Fried's estimate actually corrects a minor error in Grothendieck's original paper [Gr1] which was replicated in Ruelle's paper [Ru1].

*Remark 2.* In dimension $d = 1$ we can in fact take $\delta = \alpha = \gamma^{1/k}$, where $\gamma$ is related (by Lemma 4 in Appendix B) to the contraction ratios of the contraction maps in the iterated function scheme (see §6.3 for more details).

**3.6 The approximating functions $\Delta_N$.**

Grothendieck's expression (3.27) for the power series coefficients of the Fredholm determinant was useful for estimating their asymptotic decay rate (Proposition 7), and can be used to give rigorous estimates on dimension estimates (see §6).

However there is another expression for the $d_N(s)$, in terms of fixed points of our iterated function scheme (which correspond to periodic points of any associated expanding map, see §1). This expression arises by applying the series expansion for exp to the trace formula representation (3.26) of $\det(I - z\mathcal{L}_s)$, and then regrouping powers of $z$. Explicitly we have

**Proposition 8.** *Let $det(I - z\mathcal{L}_s) = 1 + \sum_{N=1}^{\infty} d_N(s)z^N$ be the power series expansion of the Fredholm determinant of the transfer operator $\mathcal{L}_s$. Then*

$$d_N(s) = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N}} \frac{(-1)^m}{m!}\prod_{l=1}^{m}\frac{1}{n_l}\sum_{\underline{i}\in\mathrm{Fix}(n_l)}\frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}, \tag{3.30}$$

*where the summation is over all ordered $m$-tuples of positive integers whose sum is $N$.*

The above formula (3.30) allows an *explicit* calculation of any coefficient $d_N(s)$, provided we can locate all fixed points $\cup_{n=1}^{N}\mathcal{F}ix(n)$ of compositions of $\le N$ contractions (or, equivalently, all points of period $\le N$ if our system is related to an expanding map $T$, see §1).

With this in mind, we are now in a position to define the sequence of functions $\Delta_N$ which allow us to approximate the dimension $\dim(\Lambda)$ of the limit set of our iterated function scheme. In view of Proposition 6, these functions will arise by taking the $N^{th}$ order polynomial truncation of the power series expansion for $\det(I - z\mathcal{L}_s)$, and then setting $z = 1$.

*Definition.* Define the function $\Delta_N : \mathbb{C} \to \mathbb{C}$ by

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} d_n(s) = 1 + \sum_{n=1}^{N}\sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=n}} \frac{(-1)^m}{m!}\prod_{l=1}^{m}\frac{1}{n_l}\sum_{\underline{i}\in\mathrm{Fix}(n_l)}\frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

Note we have that

$$\Delta_{N+1}(s) = \Delta_N(s) + \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N+1}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\underline{i}\in \mathrm{Fix}(n_l)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

If our iterated function scheme is related to an expanding map $T$, and Markov map $\hat{T}$, then the formula for the $\Delta_N$ is simply

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=n}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\hat{T}^{n_l}z=z} \frac{|D\hat{T}^{n_l}(z)|^{-s}}{\det\left(I - \left[D\hat{T}^{n_l}(z)\right]^{-1}\right)}.$$

*Examples.* We can easily compute the first few terms in this expansion. Let us consider the case corresponding to a full shift on 2 symbols.

(a) There are two fixed points $\hat{T}(z_0) = z_0$ and $\hat{T}(z_1) = z_1$ and we can write

$$d_1(s) = -\frac{|D\hat{T}(z_0)|^{-s}}{\det(I - [D\hat{T}(z_0)]^{-1})} - \frac{|D\hat{T}(z_1)|^{-s}}{\det(I - [D\hat{T}(z_1)]^{-1})}.$$

(b) There is a single orbit $\{z_{01}, z_{10}\}$ of prime period 2 and we can write

$$\begin{aligned} d_2(s) = {} & \frac{1}{2}\left(\frac{|D\hat{T}(z_0)|^{-s}}{\det(I - [D\hat{T}(z_0)]^{-1})}\right)^2 + \frac{1}{2}\left(\frac{|D\hat{T}(z_1)|^{-s}}{\det(I - [D\hat{T}(z_1)]^{-1})}\right)^2 \\ & + \frac{|D\hat{T}(z_0)|^{-s}|D\hat{T}(z_1)|^{-s}}{\det(I - [D\hat{T}(z_0)]^{-1})\det(I - [D\hat{T}(z_1)]^{-1})} - \frac{|D\hat{T}^2(z_{01})|^{-s}}{\det(I - [D\hat{T}^2(z_{01})]^{-1})} \\ & - \frac{1}{2}\frac{|D\hat{T}^2(z_0)|^{-s}}{\det(I - [D\hat{T}^2(z_0)]^{-1})} - \frac{1}{2}\frac{|D\hat{T}^2(z_1)|^{-s}}{\det(I - [D\hat{T}^2(z_1)]^{-1})}. \end{aligned}$$

Our most general result is the following.

**Theorem 4.** (The general theorem) *Let $\Lambda$ be the limit set for a real-analytic conformal iterated function scheme. Suppose $\Lambda$ lies in a $d$-dimensional real-analytic manifold. Then there exists $C > 0$ and $0 < \delta < 1$ such that if $0 \le s_N \le d$ is the largest real zero for $\Delta_N$ then*

$$|dim(\Lambda) - s_N| \le C\delta^{N^{1+1/d}}$$

*Proof.* As described in §1 we may assume that the manifold in question is actually $\mathbb{R}^d$, and then use the results already proved in §3.

Now $s_\infty := \dim(\Lambda)$ is the largest real zero of $\Delta(s) = \det(I - \mathcal{L}_s)$, by Proposition 6. Let $s_N$ denote the largest real zero of $\Delta_N(s)$. The definition of $\Delta_N$ ensures $s_N \to s_\infty$.

By the mean value theorem there exists $t_N$ between $s_N$ and $s_\infty$ such that

$$\begin{aligned}(s_N - s_\infty)\Delta'_N(t_N) &= \Delta_N(s_N) - \Delta_N(s_\infty) = -\Delta_N(s_\infty)\\ &= \Delta(s_\infty) - \Delta_N(s_\infty) = \sum_{n=N+1}^{\infty} d_n(s_\infty).\end{aligned}$$

If we can show that $\Delta'(s_\infty) > 0$, then $\Delta'_N(t_N)$ will be bounded away from zero for $N$ sufficiently large. Using the bound $|d_n(s)| \le C\delta^{n^{1+1/d}}$ of Proposition 7 we will then have

$$\begin{aligned}|s_N - s_\infty| &\le \left| \frac{1}{\Delta'_N(t_N)} \sum_{n=N+1}^{\infty} d_n(s_\infty) \right| \le \frac{1}{|\Delta'_N(t_N)|} \sum_{n=N+1}^{\infty} C\delta^{n^{1+1/d}}\\ &= O(\delta^{N^{1+1/d}}),\end{aligned}$$

as required.

To show that indeed $\Delta'(s_\infty) > 0$, first note that by Proposition 5 we have the infinite product representation

$$D(z) = \det(I - z\mathcal{L}_s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)),$$

where $\lambda_r(s)$ are the eigenvalues of $\mathcal{L}_s$, listed according to algebraic multiplicity, and ordered so that their absolute values are non-increasing. For real values of $s$ the leading eigenvalue $\lambda_1(s)$ is simple and positive, so $\lambda_1(s) > |\lambda_r(s)|$ for all $r \ge 2$.

Setting $Q(s) = \prod_{r=2}^{\infty}(1 - \lambda_r(s))$ we have

$$\Delta(s) = (1 - \lambda_1(s))Q(s),$$

so that

$$\Delta'(s) = -\lambda'_1(s)Q(s) + (1 - \lambda_1(s))Q'(s).$$

We know that $\lambda_1(s_\infty) = 1$, so

$$\Delta'(s_\infty) = -\lambda'_1(s_\infty)Q(s_\infty).$$

Moreover, $\lambda_1(s) = e^{p(s)}$ is a strictly decreasing analytic function of $s$, so in particular $-\lambda'_1(s_\infty) > 0$. Therefore we need to show that $Q(s_\infty) > 0$.

For any real value of $s$ (and in particular $s = s_\infty$), the coefficients in the power series expansion of $D(z)$ are all real, by (3.30). Therefore non-real zeros of $D$ arise as conjugate pairs, both with the same multiplicity. Multiplying out those factors in the product representation of $Q$ corresponding to conjugate pairs, we see that $Q(s_\infty)$ is an infinite product of strictly positive terms (since $|\lambda_r(s_\infty)| < 1$ for each $r \ge 2$). The sequence of terms converges to 1, since $|\lambda_r(s_\infty)| \to 0$. Therefore the infinite product converges to a strictly positive value. That is, $Q(s_\infty) > 0$.

*Proof of Theorem 3.* This follows immediately from Theorem 4, using the Markov partition and inverse branches of $T$ to construct an iterated function scheme (see §1).

*Remark.* For any positive integer $q$ we can consider the Fredholm determinant $\det(I - z^q\mathcal{L}_s^q)$, which factors as the product $\prod_\omega \det(I - \omega z\mathcal{L}_s)$ over all $q^{th}$ roots of unity $\omega$. Setting $z = 1$ we see that $\det(I - \mathcal{L}_s)$ is a factor of $\det(I - \mathcal{L}_s^q)$, and as before we can show that the largest real zero of $\det(I - \mathcal{L}_s^q)$ is again the Hausdorff dimension $\dim(\Lambda)$.

Thus, by analogy with the definition of the $\Delta_N$, we may define a sequence $\Delta_{rq}^{(q)}$, $r = 1, 2, \ldots$ of approximations to $\det(I - \mathcal{L}_s^q)$ as follows:

$$\Delta_{rq}^{(q)} = 1 + \sum_{j=1}^{r} \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\ldots+n_m=j}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{qn_l} \sum_{\underline{i}\in \mathrm{Fix}(qn_l)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

To compute $\Delta_{rq}^{(q)}$ we need only work with those points whose period is a multiple of $q$, up to and including $rq$. As in Theorem 4, we can show there exists $0 < \delta < 1$ such that $|\dim(\Lambda) - s_N^{(q)}| = O(\delta^{N^{1+1/d}})$, where $s_N^{(q)}$ denotes the largest real zero of $\Delta_{rq}^{(q)}$.

In some cases it is advantageous to make a judicious choice of $q$ and use the sequence of functions $\Delta_{rq}^{(q)}$ rather than the sequence $\Delta_N$ (see §7 where $\Delta_{20}^{(2)}$ gives a better empirical dimension estimate than $\Delta_{20}$).

## 4. Schottky groups and Quasifuchsian groups

The most general definition of a *classical Schottky group* is the following (see [Mas]). Let $D_1, \ldots, D_p, D_{p+1}, \ldots, D_{2p}$ be $2p$ closed Euclidean discs of dimension $d-1$, lying in $\mathbb{R}^d$ with pairwise disjoint interiors. Let $g_1, \ldots, g_p$ be Möbius maps such that each $g_i$ exchanges the interior of $D_i$ with the exterior of $D_{p+i}$ (i.e. $g_i(int(D_i)) = ext(D_{p+i})$). The corresponding classical Schottky group $\Gamma$ is the Kleinian group generated by $g_1, \ldots, g_p$.

For our theory to apply, we will always make the additional assumption that the discs themselves (rather than just their interiors) are pairwise disjoint (this corresponds to $\Gamma$ being cocompact). In this case the associated limit set $\Lambda$ of $\Gamma$ is a Cantor subset of the union of the interiors of the discs $D_1, \ldots, D_{2p}$. The same limit set can also be generated by changing $\Gamma$ to a group generated by reflections in a finite number of circles [PS]. A *reflection group* is a classical Schottky group such that $D_i = D_{p+i}$ for all $i = 1, \ldots, p$.

Classical Schottky groups are most commonly considered in dimension $d = 2$. There is a more general notion of (non-classical) Schottky groups, where the circles bounding the discs $D_1, \ldots, D_{2p}$ are replaced by Jordan curves.

A *quasifuchsian group* $\Gamma$ is a Kleinian group whose domain of discontinuity consists of two invariant components $\Omega_1$, $\Omega_2$, such that $\Omega_1/\Gamma$ is a finite Riemann surface [Mas, IX.B.2]. Every quasifuchsian group can be obtained by a quasiconformal deformation of a Fuchsian group ([Mas, IX.F]). The limit set $\Lambda$ of a quasifuchsian group is a simple closed curve.

*Proof of Theorem 1.* First suppose $\Gamma$ is a Schottky group. We define a map $T$ on the union $\cup_{j=1}^{2p} D_j$ by $T|_{int(D_j)} = g_j$ and $T|_{int(D_{p+j})} = g_j^{-1}$, for $j = 1, \ldots, p$, A Markov partition for this map just consists of the collection of interiors $\{int(D_i)\}_{i=1}^{2p}$. The corresponding

$2p \times 2p$ transition matrix $A$ has entries $A(i, p+i) = 0 = A(p+i, i)$ for each $i = 1, \ldots, p$, and all other entries are 1 (in the reflection group case, the transition matrix has zeros along the leading diagonal, and 1's elsewhere).

Now $T$ is not quite an expanding map, since the conformal derivative $|Dg_j(z)| = 1$ on the boundary of $D_j$. However, the second iterate of $T$ *is* expanding. Conformality and real-analyticity are clearly satisfied, so by Theorem 3 we deduce the result for Schottky groups.

Suppose $\Gamma$ is quasifuchsian, with limit set $\Lambda$. Now $\Gamma$ is quasi-conformally conjugate to some Fuchsian group $\Gamma'$ ([Mas, IX.F]). Bowen & Series [BS] proved there exists an expanding Markov map $S : S^1 \to S^1$ which faithfully models the action of $\Gamma'$, and the quasiconformal deformation conjugates this to an expanding Markov map $T : \Lambda \to \Lambda$. Conformality and real-analyticity are clearly satisfied, so the quasifuchsian case of Theorem 1 follows from the general Theorem 3.

## 5. Julia sets

**5.1 Markov partitions and hyperbolicity.**

For a holomorphic map $f : U \to U$ defined on some domain $U \subset \mathbb{C}$ of the complex plane, we define its *Julia set* $\mathcal{J}$ to be the closure of the repelling periodic points of $f$. More precisely,

$$\mathcal{J} = \overline{\cup_{n \geq 1} \{z \in \mathbb{C} : f^n z = z \text{ and } |(f^n)'(z)| > 1\}}.$$

The Julia set is a closed $f$-invariant set, and is locally maximal. Clearly $f : \mathcal{J} \to \mathcal{J}$ is conformal and real-analytic, so *if* $f : \mathcal{J} \to \mathcal{J}$ is also Markov and hyperbolic then we can use Theorem 3 to compute the Hausdorff dimension $\dim(\mathcal{J})$.

**Lemma 3.** *Markov partitions always exist for expanding maps $f : \mathcal{J} \to \mathcal{J}$* [Ru2, p.146], [Bo1].

The following well-known result (see [St, p. 118]) characterizes hyperbolic rational maps.

**Proposition 9.** *The following are equivalent*

1. *$f : \mathcal{J} \to \mathcal{J}$ is hyperbolic;*
2. *$\mathcal{J}$ is disjoint from the orbit of the critical points $\mathcal{C} = \{z : f'(z) = 0\}$ (i.e. $\mathcal{J} \cap (\cup_{n=0}^{\infty} f^n(\mathcal{C})) = \emptyset$)*

If we restrict to the case of quadratic polynomials $f_c(z) = z^2 + c$, then there is the following characterisation of hyperbolicity (see [CG, p128]) in terms of the *Mandelbrot set* $\mathcal{M} := \{c \in \mathbb{C} : |f_c^n(0)| \not\to +\infty \text{ as } n \to +\infty\}$.

**Proposition 10.** *Let $f_c(z) = z^2 + c$ be a quadratic map, with Julia set $\mathcal{J}_c$. Then $f_c : \mathcal{J}_c \to \mathcal{J}_c$ is hyperbolic if and only if either $c$ lies outside $\mathcal{M}$, or $f_c$ has an attracting periodic point $z$ (i.e. $f_c^n z = z$ for some $n$, and $|(f^n)'(z)| < 1$).*

Note that if $c \notin \mathcal{M}$ then the Julia set $\mathcal{J}_c$ is a Cantor set, while if $c \in \mathcal{M}$ then $\mathcal{J}_c$ is connected.

The interior $\mathrm{int}(\mathcal{M})$ of the Mandelbrot set is a union of simply connected components, the largest of which is the *main cardioid* $\mathcal{M}_1 = \{w \in \mathbb{C} : |1 - \sqrt{1 - 4w}| < 1\}$. For any

$c \in \mathcal{M}_1$ the map $f_c$ has the attracting fixed point $\frac{1}{2}(1-\sqrt{1-4w})$, so is hyperbolic by Proposition 9.

More generally, a component $\mathcal{H}$ of $\mathrm{int}(\mathcal{M})$ contains a parameter $c$ for which $f_c$ is hyperbolic if and only if $f_{c'}$ is hyperbolic for *every* $c' \in \mathcal{H}$. We then call $\mathcal{H}$ a *hyperbolic component* [St, p.160]. It is unknown whether every component of $\mathrm{int}(\mathcal{M})$ is hyperbolic.

**5.2 Trace formula for hyperbolic holomorphic Markov maps.**

In §3.4 we proved, in a very general case, a trace formula for iterates of the transfer operator $\mathcal{L}_s$ in terms of fixed points of an iterated function scheme. Now we use this to derive a simplified formula in the special case of hyperbolic holomorphic Markov maps restricted to their Julia sets.

**Proposition 11.** *Let $f : \mathcal{J} \to \mathcal{J}$ be a hyperbolic holomorphic Markov map, with Julia set $\mathcal{J}$. Let $\mathcal{L}_s$ be the associated transfer operator. Then*

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{\substack{f^n z = z \\ z \in \mathcal{J}}} |(f^n)'(z)|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}((f^n)'(z))}{|(f^n)'(z)|^2}\right)^{-1}.$$

*Proof.* Identifying $\mathbb{R}^2$ with $\mathbb{C}$ we think of $\mathcal{J}$ as lying in the real section $\mathbb{R}^2$ of $\mathbb{C}^2$. Letting $T(x,y) = (\mathrm{Re}\,(f(x+iy))\,, \mathrm{Im}\,(f(x+iy)))$, we know that $T$ extends to a holomorphic map on some open neighbourhood $U \subset \mathbb{C}^2$ of $\mathcal{J}$.

Using the Markov partition $X_1, \ldots, X_k$, we define a collection of inverse branches $\{S_j\}_{j=1}^k$ to $T$ satisfying $T \circ S_j = id|_{TX_j}$. As in §1, we consider the set of all admissible length-$n$ strings $\underline{j} = (j_1, \ldots, j_n)$, and note that they correspond bijectively to those length-$(n+1)$ strings in the set $\mathrm{Fix}(n)$. For the purposes of this proof let $\mathcal{L}_{s,\underline{j}}$ denote the operator $\mathcal{L}_{s,\underline{i}}$ (see (3.19)), where $\underline{i} = (j_n, j_1, \ldots, j_n)$. By Proposition 3 we therefore have that $\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|\underline{j}|=n} \mathrm{tr}(\mathcal{L}_{s,\underline{j}})$, where

$$\mathrm{tr}(\mathcal{L}_{s,\underline{j}}) = \frac{|(f^n)'(z_{\underline{j}})|^{-s}}{\det(I - DS_{\underline{j}}(x_{\underline{j}}, y_{\underline{i}}))}. \tag{5.1}$$

Here $S_{\underline{j}}$ is as in §1 (i.e. an inverse branch of $T^n$), and $(x_{\underline{j}}, y_{\underline{j}})$ denotes the unique fixed point of $S_{\underline{j}}$. Note that $(x_{\underline{j}}, y_{\underline{j}}) \in \mathbb{R}^2$, and we let $z_{\underline{j}} = x_{\underline{j}} + iy_{\underline{j}}$.

The numerator in (5.1) follows immediately from Proposition 3, since the conformal derivative $|DS_{\underline{j}}(x_{\underline{j}}, y_{\underline{j}})| = |DT^n(x_{\underline{j}}, y_{\underline{j}})|^{-1} = |f^n(x_{\underline{j}} + iy_{\underline{j}})|^{-1}$.

To treat the denominator in (5.1) we note that

$$\det(I - DS_{\underline{j}}(x_{\underline{j}}, y_{\underline{i}})) = \frac{\det(DT^n(x_{\underline{j}}, y_{\underline{j}}) - I)}{\det(DT^n(x_{\underline{j}}, y_{\underline{j}}))}. \tag{5.2}$$

Let us write $T^n(x,y) = ((T^n)_1(x,y), (T^n)_2(x,y))$, so that

$$\begin{aligned} DT^n(x_{\underline{j}}, y_{\underline{j}}) &= \begin{pmatrix} D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) & D_2(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) \\ D_1(T^n)_2(x_{\underline{j}}, y_{\underline{j}}) & D_2(T^n)_2(x_{\underline{j}}, y_{\underline{j}}) \end{pmatrix} \\ &= \begin{pmatrix} D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) & D_2(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) \\ -D_2(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) & D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) \end{pmatrix}, \end{aligned}$$

by the Cauchy-Riemann equations, where $D_i$ denotes partial differentiation with respect to the $i^{th}$ coordinate.

Therefore

$$\begin{aligned}\det(DT^n(x_{\underline{j}}, y_{\underline{j}}) - I) &= (1 - D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}))^2 + (D_2(T^n)_1(x_{\underline{j}}, y_{\underline{j}}))^2 \\ &= 1 - 2D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) + (D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}))^2 + (D_2(T^n)_1(x_{\underline{j}}, y_{\underline{j}}))^2 \\ &= 1 - 2D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) + \det(DT^n(x_{\underline{j}}, y_{\underline{j}})). \qquad (5.3)\end{aligned}$$

Now

$$D_1(T^n)_1(x_{\underline{j}}, y_{\underline{j}}) = \mathrm{Re}((f^n)'(z_{\underline{j}})), \tag{5.4}$$

and area considerations mean that

$$\det(DT^n(x_{\underline{j}}, y_{\underline{j}})) = |(f^n)'(z_{\underline{j}})|^2, \tag{5.5}$$

so substituting (5.4), (5.5) into (5.3), and then into (5.2) gives us

$$\det(I - DS_{\underline{j}}(x_{\underline{j}}, y_{\underline{j}})) = 1 + \frac{1 - 2\mathrm{Re}((f^n)'(z_{\underline{i}}))}{|(f^n)'(z_{\underline{j}})|^2}.$$

Substituting into (5.1) gives

$$\mathrm{tr}(\mathcal{L}_{\underline{j}}) = |(f^n)'(z_{\underline{j}})|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}((f^n)'(z_{\underline{j}}))}{|(f^n)'(z_{\underline{j}})|^2}\right)^{-1},$$

then using the fact that $\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|\underline{j}|=n} \mathrm{tr}(\mathcal{L}_{s,\underline{j}})$ gives the result.

In particular, for hyperbolic quadratic maps $f_c(z) = z^2 + c$ we have the following trace formula.

**Corollary 11.1.** *Let $f_c(z) = z^2 + c$ be a quadratic map, with hyperbolic Julia set $\mathcal{J}_c$. Let $\mathcal{L}_s$ be the associated transfer operator. Then*

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{\substack{f_c^n z = z \\ z \in \mathcal{J}_c}} 2^{-sn} |\pi(z)|^{-s} \left(1 + \frac{1 - 2^{n+1}\mathrm{Re}(\pi(z))}{2^{2n}|\pi(z)|^2}\right)^{-1},$$

*where $\pi(z) := \prod_{r=0}^{n-1} f_c^r(z)$ for period-$n$ points $z$.*

**Corollary 11.2.** *Let $f : \mathcal{J} \to \mathcal{J}$ be a hyperbolic holomorphic map, with Julia set $\mathcal{J}$. The corresponding functions $\Delta_N$, whose leading zeros $s_N$ give a sequence of approximations to dim($\mathcal{J}$), are given by the formula*

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m=n}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\substack{f^{n_l} z = z \\ z \in \mathcal{J}}} |(f^{n_l})'(z)|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}((f^{n_l})'(z))}{|(f^{n_l})'(z)|^2}\right)^{-1}.$$

*Proof of Theorem 2.* This follows from Theorem 3 upon setting $d = 2$, and using Corollary 11.2 to express the functions $\Delta_N$ in terms of derivatives at periodic points.

## 6. Fuchsian reflection in three symmetric circles

**6.1 McMullen's three circle family.**

We now analyse in detail one of the examples considered by McMullen [McM3], a family of symmetric Fuchsian reflection groups.

Let $C_0, C_1, C_2$ be three circles of equal radii arranged symmetrically around the unit circle $S^1$, each intersecting $S^1$ orthogonally, and meeting $S^1$ in an arc of length $\theta$. Let $\theta$ denote the angle at the origin subtended by the two points of intersection.

We do not want the $C_i$ to intersect each other, so we demand that $0 < \theta < 2\pi/3$. For definiteness let us suppose each $C_i$ has radius $r = r_\theta = \tan\frac{\theta}{2}$, and that the circle centres are at the points $z_0 = a$, $z_1 = ae^{2\pi i/3}$ and $z_2 = ae^{-2\pi i/3}$, where

$$a = a_\theta = \sqrt{1+r^2} = \sec\frac{\theta}{2}.$$

Since $0 < \theta < 2\pi/3$, we see that

$$0 < r_\theta < \sqrt{3} \text{ and } 0 < a_\theta < 2.$$

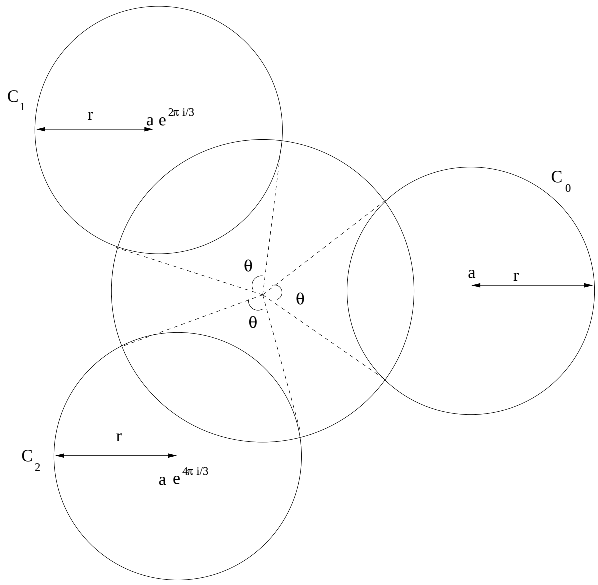


Figure 5. Reflection in three circles

The reflection $\rho_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the circle $C_i$ takes the explicit form

$$\begin{aligned} \rho_i(z) &= \frac{r^2}{\bar{z} - \bar{z}_i} + z_i \\ &= \frac{r^2}{|z - z_i|^2}(z - z_i) + z_i. \end{aligned}$$

Note these reflections are *anti-holomorphic*, and in particular *reverse* orientation.

Let $\Gamma_\theta$ be the group generated by these three reflections, and $\Lambda_\theta$ the associated limit set.

Since the reflections $\rho_i$ each preserve the unit circle $S^1$, then $\Lambda_\theta \subset S^1$. However, by a conformal change of coordinates we will transform the limit set to a subset of the real line, and by Theorem 3 with $d = 1$ our algorithm will give $O(\delta^{N^2})$ convergence to $\dim(\Lambda_\theta)$.

More precisely, let

$$f(z) = i\frac{z-1}{z+1}$$

be our conformal change of coordinates. Thus if we consider the restriction of each $\rho_j$ to $S^1$, then the conjugated map $T_j := f \circ \rho_j|_{S^1} \circ f^{-1}$ is a map of the real line to itself.

A calculation shows that $T_j : \mathbb{R} \to \mathbb{R}$ is the Möbius map

$$T_j(z) = \frac{i(z_j - \bar{z}_j)z + z_j + \bar{z}_j - 2}{(z_j + \bar{z}_j + 2)z + i(\bar{z}_j - z_j)}.$$

Symmetry considerations mean we have

$$T_0(z) = \frac{a_\theta - 1}{(a_\theta + 1)z} = \frac{\sec\frac{\theta}{2} - 1}{(\sec\frac{\theta}{2} + 1)z},$$

$$T_1(z) = \frac{-\sqrt{3}a_\theta z - (2 + a_\theta)}{(2 - a_\theta)z + \sqrt{3}a_\theta},$$

$$T_2(z) = \frac{\sqrt{3}a_\theta z - (2 + a_\theta)}{(2 - a_\theta)z - \sqrt{3}a_\theta}.$$

Geometrically, each $T_j : \mathbb{R} \to \mathbb{R}$ is inversion in some interval. $T_0$ is inversion in the interval $I_0 = (-\tan\frac{\theta}{4}, \tan\frac{\theta}{4})$. $T_1$ and $T_2$ are inversions in the intervals $I_1 = (\tan(\frac{\pi}{3} - \frac{\theta}{4}), \tan(\frac{\pi}{3} + \frac{\theta}{4}))$ and $I_2 = (-\tan(\frac{\pi}{3} + \frac{\theta}{4}), -\tan(\frac{\pi}{3} - \frac{\theta}{4}))$ respectively.

*Remark.* Since our change of coordinates $f$ is bi-Lipschitz, the Hausdorff dimension of the limit set generated by the $T_j$ is the same as that generated by our initial reflection system given by the $\rho_j$.

**6.2 Compositions and weight functions.**

Now consider the absolute values of the derivatives of the maps $T_0, T_1, T_2$. We have

$$v_0(z) := |T_0'(z)| = \frac{a_\theta - 1}{(a_\theta + 1)z^2}$$

Now $v_0 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is real analytic on any interval not containing zero. It extends to a map $v_0 : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, holomorphic on any domain not containing zero. For any real $s \geq 0$, the weight function

$$\begin{aligned} w_{s,0}(z) := v_0(z)^s &= \left(\frac{a_\theta - 1}{(a_\theta + 1)z^2}\right)^s \\ &= \exp\left(s \log \frac{a_\theta - 1}{(a_\theta + 1)z^2}\right) \end{aligned}$$

is holomorphic on any cut-plane $\mathbb{C} \setminus l$, where $l$ is a half-line emanating from the point $y_0 = 0$. In particular, $w_{s,0}$ is holomorphic on both $\mathbb{C} \setminus (-\infty, 0]$ and $\mathbb{C} \setminus [0, \infty)$.

A similar analysis applies to the maps $v_1 : \mathbb{R} \setminus \{y_1\} \to \mathbb{R}$ and $v_2 : \mathbb{R} \setminus \{y_2\} \to \mathbb{R}$, where

$$v_1(z) := |T_1'(z)| = \frac{4(a_\theta^2 - 1)}{\left((2-a_\theta)z + \sqrt{3}a_\theta\right)^2} = \frac{4r_\theta^2}{\left((2-a_\theta)z + \sqrt{3}a_\theta\right)^2}$$

and

$$v_2(z) := |T_2'(z)| = \frac{4(a_\theta^2 - 1)}{\left((2-a_\theta)z - \sqrt{3}a_\theta\right)^2} = \frac{4r_\theta^2}{\left((2-a_\theta)z - \sqrt{3}a_\theta\right)^2}$$

and where $y_1 = -\sqrt{3}a_\theta/(2-a_\theta)\}$, $y_2 = \sqrt{3}a_\theta/(2-a_\theta)\}$.

Having initially restricted $v_1, v_2$ to appropriate subsets of the real line, we then observe that these restrictions are real-analytic. We then consider the holomorphic extensions to appropriate domains in $\mathbb{C}$. Raising to the $s^{th}$ power, $s \geq 0$, we see that the weight function

$$\begin{aligned} w_{s,1}(z) := v_1(z)^s &= \left( \frac{4r_\theta^2}{\left((2-a_\theta)z + \sqrt{3}a_\theta\right)^2} \right)^s \\ &= \exp\left( s \log\left( \frac{4r_\theta^2}{(2-a_\theta)z + \sqrt{3}a_\theta} \right)\right) \end{aligned}$$

is holomorphic on the cut-plane $\mathbb{C} \setminus (-\infty, y_1]$, where $y_1 = -\sqrt{3}a_\theta/(2-a_\theta)$. Similarly, the weight function

$$\begin{aligned} w_{s,2}(z) := v_2(z)^s &= \left( \frac{4r_\theta^2}{\left((2-a_\theta)z - \sqrt{3}a_\theta\right)^2} \right)^s \\ &= \exp\left( s \log\left( \frac{4r_\theta^2}{(2-a_\theta)z - \sqrt{3}a_\theta} \right)\right) \end{aligned}$$

is holomorphic on the cut-plane $\mathbb{C} \setminus [y_2, \infty)$, where $y_2 = \sqrt{3}a_\theta/(2-a_\theta)$.

Now clearly each $T_j$ has a holomorphic extension to the whole of $\mathbb{C}$, with a single pole at $y_j$, so by abuse of notation let us write $T_j : \mathbb{C} \to \mathbb{C}$. If $D_j$ denotes the disc which is symmetric about $\mathbb{R}$ and interects $\mathbb{R}$ in the interval $I_j$, then now $T_j$ is inversion in the circle $\partial D_j$.

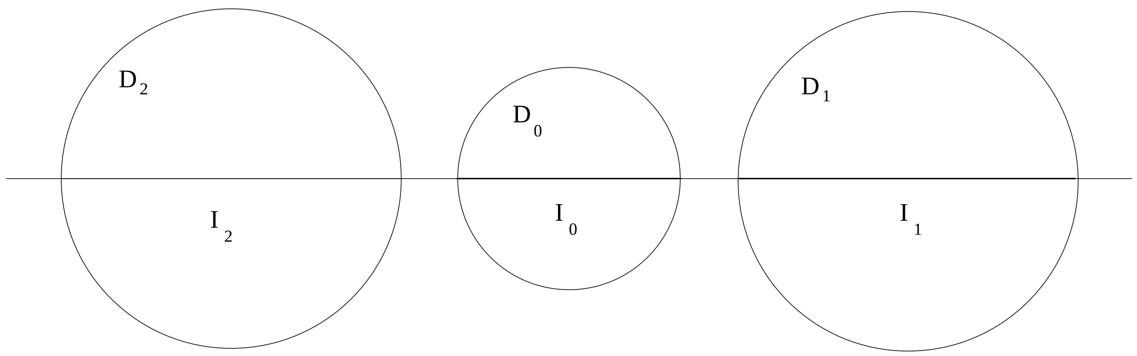


FIGURE 6. The circles of inversion in the real line

If $r_j$ denotes the radius of $D_j$, then we have

$$r_0 = \tan\frac{\theta}{4} \quad , \quad r_1 = r_2 = \frac{\tan(\frac{\pi}{3}+\frac{\theta}{4}) - \tan(\frac{\pi}{3}-\frac{\theta}{4})}{2}.$$

Note that for $0 < \theta < 2\pi/3$, the $D_j$ do not intersect each other.

Let $D = \coprod_{i=0}^{2} D_i$ denote the disjoint union of these discs. Let $\mathcal{A}_\infty(D)$ denote the space of bounded functions $h : \bar{D} \to \mathbb{C}$ whose restriction $h|_{D_j}$ is a holomorphic function $h_j : \bar{D} \to \mathbb{C}$, for each $j = 0, 1, 2$. This is a Banach space with respect to the supremum norm.

Let $E_0$ be the disc centred at 0 whose boundary intersects the real axis at the points $\pm\tan(\frac{\theta}{4} - \frac{\pi}{3})$. Let $E_1$ be the disc, concentric with $D_1$, whose boundary intersects $\mathbb{R}$ at $-\tan\frac{\theta}{4}$. Let $E_2$ be the disc, concentric with $B_2$, whose boundary intersects $\mathbb{R}$ at $\tan\frac{\theta}{4}$.

If the function $g$ is holomorphic on $D_j$, then the composition $g \circ T_j$ is holomorphic on $T_j^{-1}D_j = T_jD_j = ext(D_j)$. In particular, $g \circ T_j$ is holomorphic on both the discs $E_k$, for $k \in \{0, 1, 2\} \setminus \{j\}$.

The above discussion means that if $h = (h_0, h_1, h_2) \in \mathcal{A}_\infty(D)$, then the function

$$z \mapsto h(T_1z)w_{s,1}(z) + h(T_2z)w_{s,2}(z) = h_1(T_1z)w_{s,1}(z) + h_2(T_2z)w_{s,2}(z) \tag{6.1}$$

is holomorphic on the disc $E_0$, (since $w_{s,1}$ is holomorphic on the cut plane $\mathbb{C} \setminus (-\infty, y_1]$, $w_{s,2}$ is holomorphic on the cut plane $\mathbb{C} \setminus [y_2, \infty)$, and neither of the cuts intersects $E_0$).

The function

$$z \mapsto h(T_0z)w_{s,0}(z) + h(T_2z)w_{s,2}(z) = h_0(T_0z)w_{s,0}(z) + h_2(T_2z)w_{s,2}(z) \tag{6.2}$$

is holomorphic on the disc $E_1$, (since both $w_{s,0}$ and $w_{s,2}$ are holomorphic on the cut plane $\mathbb{C} \setminus [0, \infty)$).

Similarly, the function

$$z \mapsto h(T_0z)w_{s,0}(z) + h(T_1z)w_{s,1}(z) = h_0(T_0z)w_{s,0}(z) + h_1(T_1z)w_{s,1}(z) \tag{6.3}$$

is holomorphic on the disc $E_2$, (since both $w_{s,0}$ and $w_{s,1}$ are holomorphic on the cut plane $\mathbb{C} \setminus (-\infty, 0]$).

**6.3 The transfer operator and Fredholm determinant.**

Letting $(\mathcal{L}_su)_0(z), (\mathcal{L}_su)_1(z), (\mathcal{L}_su)_2(z)$ denote the functions defined by the formulae in (6.1), (6.2), (6.3) respectively, we can define the transfer operator $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(E)$ by

$$(\mathcal{L}_sh)(z) = \begin{cases} (\mathcal{L}_sh)_0(z) = h(T_1z)w_{s,1}(z) + h(T_2z)w_{s,2}(z) & \text{if } z \in E_0 \\ (\mathcal{L}_sh)_1(z) = h(T_0z)w_{s,0}(z) + h(T_2z)w_{s,2}(z) & \text{if } z \in E_1 \\ (\mathcal{L}_sh)_2(z) = h(T_0z)w_{s,0}(z) + h(T_1z)w_{s,1}(z) & \text{if } z \in E_2, \end{cases} \tag{6.4}$$

Let us also define the component transfer operators $\mathcal{L}_{s,j} : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(E_j)$ by $\mathcal{L}_{s,j}u = (\mathcal{L}_su)_j$.

The crucial point in our analysis is that $\mathcal{L}_s h$ is holomorphic on a *larger* domain than $h$. For $j=0,1,2$, let $r_j'$ denote the radius of the disc $E_j$. So we have

$$r_0' = \tan\left(\frac{\pi}{3} - \frac{\theta}{4}\right),$$

$$r_1' = r_2' = \frac{1}{2}\left(\tan\left(\frac{\pi}{3} + \frac{\theta}{4}\right) + \tan\left(\frac{\pi}{3} - \frac{\theta}{4}\right)\right) - \tan\frac{\theta}{4}.$$

Let $\gamma_j(\theta) = \gamma_j < 1$ denote the ratio of the radii of the two discs $D_j$, $E_j$ centred at $y_j$. We have the contraction ratios

$$\gamma_0 = \frac{r_0}{r_0'} = \frac{\tan\frac{\theta}{4}}{\tan(\frac{\pi}{3} - \frac{\theta}{4})}, \tag{6.5}$$

and

$$\begin{aligned}\gamma_1 = \gamma_2 = \frac{r_2}{r_2'} &= \frac{\frac{1}{2}\left(\tan(\frac{\pi}{3}+\frac{\theta}{4}) - \tan(\frac{\pi}{3}-\frac{\theta}{4})\right)}{\frac{1}{2}\left(\tan(\frac{\pi}{3}+\frac{\theta}{4}) + \tan(\frac{\pi}{3}-\frac{\theta}{4})\right) - \tan\frac{\theta}{4}} \\ &= \frac{\sin\frac{\theta}{2}}{\frac{\sqrt{3}}{2} - 2\tan\frac{\theta}{4}\cos(\frac{\pi}{3}+\frac{\theta}{4})\cos(\frac{\pi}{3}-\frac{\theta}{4})}.\end{aligned} \tag{6.6}$$

Note that $\gamma_0(\theta) < \gamma_1(\theta) = \gamma_2(\theta)$ for all $0 < \theta < 2\pi/3$.

*Remark.* The contraction ratios $\gamma_j$ are degenerate *cross-ratios* (where two of the points happen to coincide). Since all fractional linear maps preserve cross-ratios, then we cannot hope to obtain smaller contraction ratios by taking the coordinate change $f$ to be a different fractional linear map.

For $0 < \varepsilon < r_j'$, let $\Gamma_{j,\varepsilon}$ denote the circle centred at $y_j$, of radius $r_j' - \varepsilon$.

For each $j = 0,1,2$, and $h \in \mathcal{A}_\infty(D)$, and $z \in D$, we have

$$\begin{aligned}(\mathcal{L}_s h)_j(z) &= \frac{1}{2\pi i}\int_{\Gamma_{j,\varepsilon}} \frac{(\mathcal{L}_s h)_j(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_{j,\varepsilon}} \frac{(\mathcal{L}_s h)_j(\xi)}{\xi - y_j}\left(1 - \frac{z-y_j}{\xi - y_j}\right)^{-1} d\xi \\ &= \sum_{n=0}^{\infty} l_n^{(j)}(h) v_n^{(j)}(z),\end{aligned}$$

where

$$v_n^{(j)}(z) = (z-y_j)^n \quad \text{and} \quad l_n^{(j)}(u) = \frac{1}{2\pi i}\int_{\Gamma_{j,\varepsilon}} \frac{(\mathcal{L}_s h)_j(\xi)}{(\xi - y_j)^{n+1}} d\xi. \tag{6.7}$$

Now define the sequence of functions $v_k \in \mathcal{A}_\infty(D)$ by

$$v_{3n+j}|_{D_j} = v_n^{(j)}, \quad \text{and} \quad v_{3n+j}|_{D_i} = 0 \text{ for } i \in \{0,1,2\} \setminus \{j\},$$

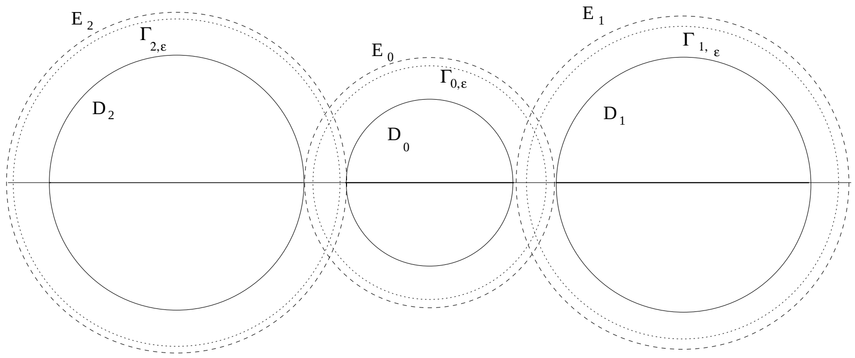


FIGURE 7. THREE PAIRS OF CONCENTRIC DISCS, AND CORRESPONDING CIRCLES OF INTEGRATION

and the sequence of linear functionals $l_k \in \mathcal{A}_\infty(D)^*$ by

$$l_{3n+j} = l_n^{(j)}.$$

Then we can write our transfer operator $\mathcal{L}_s$ as the single infinite series

$$\mathcal{L}_s h(z) = \sum_{k=0}^{\infty} l_k(h) v_k(z). \tag{6.8}$$

*Remark.* Notice that for any $j \in \{0,1,2\}$, and any integer $m \geq 0$ we have $(\mathcal{L}_s v_{3m+j})_j(z) = 0$. This implies that for any $j \in \{0,1,2\}$, and any integers $m, n \geq 0$ we have

$$l_{3n+j}(v_{3m+j}) = l_n^{(j)}(v_{3m+j}) = \frac{1}{2\pi i}\int_{\Gamma_{j,\varepsilon}} \frac{(\mathcal{L}_s v_{3m+j})_j(\xi)}{(\xi - y_j)^{n+1}} d\xi = 0.$$

Consequently, in the infinite matrix $(l_\alpha(v_\beta))_{\alpha,\beta=0}^{\infty}$, the entries in diagonals of the form $\{(3m+i,i) : i \geq 0\}$ and $\{(i, 3n+i) : i \geq 0\}$ are all zero. It is therefore possible to improve slightly the Hadamard bound (see proof of Proposition 7) on the determinant of submatrices.

As in §3.3, the infinite series expression (6.8) shows that $\mathcal{L}_s$ is nuclear of order zero, since the terms $\lambda_k = ||l_k||_\infty ||v_k||_\infty$ decay exponentially fast (here the uniform norm is taken on the disjoint union $D$).

To estimate the exponential decay rate of the $\lambda_k$ we note that for $z \in A$ we clearly have $|v_{3n+j}(z)| \leq r_j^n$. By (6.7) we also have

$$||l_{3n+j}||_\infty \leq \frac{||\mathcal{L}_{s,j}||_{\infty,E_j}}{2\pi(r_j' - \varepsilon)^{n+1}}.$$

Note in this estimate $||\cdot||_{\infty,E_j}$ denotes the uniform norm with respect to the (larger) disc $E_j$. So for all $0 < \varepsilon < r_j'$ we have the estimate

$$\lambda_{3n+j} = ||l_{3n+j}||_\infty ||v_{3n+j}||_\infty \leq \frac{||\mathcal{L}_{s,j}||_{\infty,E_j}}{2\pi(r_j' - \varepsilon)} \left(\frac{r_j}{r_j' - \varepsilon}\right)^n.$$

Letting $\varepsilon \to 0$ gives, for $j = 0, 1, 2$ and $n \geq 0$,

$$\lambda_{3n+j} = ||l_{3n+j}||_\infty ||v_{3n+j}||_\infty \leq K_j \left(\frac{r_j}{r'_j}\right)^n = K_j \gamma_j^n, \tag{6.9}$$

where

$$K_j = K_j(\theta) = \frac{||\mathcal{L}_{s,j}||_{\infty,E_j}}{2\pi r'_j}, \tag{6.10}$$

and where the $\gamma_j$ are given by (6.5), (6.6).

Having established the exponential decay rate (6.9), we now use Lemma 4 in Appendix B to combine these and find $0 < \gamma < 1$ and $K > 0$ such that for all $n \geq 0$

$$\lambda_n \leq K\gamma^n. \tag{6.11}$$

The exponential decay rate (6.11) corresponds to (3.21) with $d = 1$, and where $\gamma$ now denotes $\alpha$.

Now we proceed as in the proof of Proposition 6 to estimate the decay rate of the power series coefficients $d_N(s)$ of the Fredholm determinant $\det(I - z\mathcal{L}_s)$. Since $d = 1$ we have the Euler formula [E]

$$\sum_{k_1 < \ldots < k_N} \gamma^{k_1 + \ldots + k_N} = \frac{\gamma^{N(N-1)/2}}{(1-\gamma)(1-\gamma^2)\ldots(1-\gamma^N)},$$

which is proved by successively summing the geometric series on the lefthand side. From (3.28) we then have

$$|d_N(s)| \leq \frac{K^N N^{N/2} \gamma^{N(N-1)/2}}{(1-\gamma)\ldots(1-\gamma^N)}. \tag{6.12}$$

**6.4 Estimates on transfer operator norms.**

Now we would like to estimate the norms $||\mathcal{L}_{s,j}||_{\infty,A'_j}$. This will be particularly important for establishing *rigorous* dimension estimates, such as in §7, where we want to estimate the constants $K_j$ defined by (6.10). By (6.2), (6.3), (6.4) defining the $\mathcal{L}_{s,j}$ we clearly have

$$||\mathcal{L}_{s,j}||_{\infty,E_j} \leq \sum_{\substack{k=0 \\ k \neq j}}^{2} ||w_s^{(k)}||_{\infty,E_j}$$

for each $j = 0, 1, 2$.

For $j = 0$ we have $||w_{s,1}||_{\infty,E_0} = 1$, this value of $|w_{s,1}(z)|$ being attained uniquely at the point $z = -\tan(\frac{\theta}{4} - \frac{\pi}{3})$ where $\overline{E_0}$ intersects the isometric circle $\partial D_1$ of the map $T_1$. Similarly we have $||w_{s,2}||_{\infty,E_0} = 1$, this value of $|w_{s,2}(z)|$ being attained uniquely at the point $z = \tan(\frac{\theta}{4} - \frac{\pi}{3})$ where $\overline{E_0}$ intersects the isometric circle $\partial D_2$ of the map $T_2$. Therefore we have the bound

$$||\mathcal{L}_{s,0}||_{\infty,E_0} \leq 2. \tag{6.13}$$

For $j=1$ we see that $||w_{s,0}||_{\infty,E_1}=1$, this value of $|w_{s,0}(z)|$ being attained uniquely at the point $z=-\tan\frac{\theta}{4}$ where $\overline{E_1}$ intersects the isometric circle $\partial D_0$ of the map $T_0$. We have that

$$||w_{s,2}||_{\infty,E_1}=w_{s,2}\left(-\tan\frac{\theta}{4}\right)=\left(\frac{4r_\theta^2}{\left(-(2-a_\theta)\tan\frac{\theta}{4}-\sqrt{3}a_\theta\right)^2}\right)^s.$$

Note in particular that $-\tan\frac{\theta}{4}$ lies strictly outside the isometric circle of $T_2$, so (for $s>0$) we have $||w_{s,2}||_{\infty,E_1}<1$.

Therefore we have

$$||\mathcal{L}_{s,1}||_{\infty,E_1}\leq 1+\left(\frac{4r_\theta^2}{\left(-(2-a_\theta)\tan\frac{\theta}{4}-\sqrt{3}a_\theta\right)^2}\right)^s. \tag{6.14}$$

A similar argument for the case $j=2$ gives us

$$||\mathcal{L}_{s,2}||_{\infty,E_2}\leq 1+w_{s,1}\left(\tan\frac{\theta}{4}\right)=1+\left(\frac{4r_\theta^2}{\left((2-a_\theta)\tan\frac{\theta}{4}+\sqrt{3}a_\theta\right)^2}\right)^s. \tag{6.15}$$

(Similarly to before, note that $\tan\frac{\theta}{4}$ lies strictly outside the isometric circle of $T_1$, so (for $s>0$) we have $||w_{s,1}||_{\infty,E_2}<1$.)

## 7. Rigorous dimension estimates for limit sets

In this section we consider a particular choice of the parameter $\theta$ in our family of Fuchsian reflection groups. We choose $\theta=\pi/6$, and strive for the best possible rigorous estimate of $\dim(\Lambda_\theta)$. We use the sequence $\Delta_N(s)$ of approximations to $\det(I-\mathcal{L}_s)$, and also the sequence $\Delta_{2r}^{(2)}(s)$ of approximations to $\det(I-\mathcal{L}_s^2)$ (cf. the Remark at the end of §3.6).

*Example.* $\theta=\pi/6$.

For $\theta=\pi/6$, the various radii $r_j$ and $r_j'$ are

$$\begin{aligned}
r_0&=\tan\pi/24=0.13165\ldots\\
r_0'&=\tan 7\pi/24=1.30322\ldots\\
r_1=r_2&=\frac{\tan\frac{9\pi}{24}-\tan\frac{7\pi}{24}}{2}=0.55549\ldots\\
r_1'=r_2'&=\frac{1}{2}\left(\tan\frac{9\pi}{12}+\tan\frac{7\pi}{12}\right)-\tan\frac{\pi}{24}=1.72706\ldots
\end{aligned}$$

Defining the contraction ratios $\gamma_i=r_i/r_i'$, we have $\gamma_0=0.10102\ldots$ and $\gamma_1=\gamma_2=0.32164\ldots$

For $s\geq 0$ the bounds (6.13), (6.14) and (6.15) give us $||\mathcal{L}_{s,j}||_{\infty,E_j}\leq 2$ for each $j=0,1,2$.

*Remark.* The bounds for $j = 1$ and 2 could be sharpened by obtaining a lower bound on $s = \dim(\Lambda)$. For example, such a lower bound is given by $-\log 3/\log\lambda$, where $\lambda = \max_{0\le j\le 2}\max_{z\in\Lambda}|T_j'(z)|$ is the slowest contraction rate on the limit set $\Lambda$ (see [Fa1], Prop. 9.7). Of course better lower bounds are obtainable from our algorithm itself, so that having used the algorithm once with some crude lower bound on $\dim(\Lambda)$ giving the estimates on $||\mathcal{L}_{s,j}||_{\infty,E_j}$, we can then go back and use the improved lower bound to sharpen the estimates on $||\mathcal{L}_{s,j}||_{\infty,E_j}$, and continue bootstrapping in this way to obtain successively better estimates on $\dim(\Lambda)$.

Using (6.10) we then have

$$K_j = K_j(\pi/6) = \frac{||\mathcal{L}_{s,j}||_{\infty,E_j}}{2\pi r_j'},$$

so that $K_0 \le 0.41482\ldots$ and $K_1 = K_2 \le 0.54974\ldots$.

| $N$ | Largest zero of $\Delta_N$ |
|---|---|
| 2 | 0.146334812960077410554547484014545 96 |
| 3 | 0.18423440272351767688822531747382350 |
| 4 | 0.18399977929621235204864644797773486 |
| 5 | 0.18398305039516509087579859265399133 |
| 6 | 0.18398305988417009403195596234810316 |
| 7 | 0.18398306122261622100816402885866734 |
| 8 | 0.18398306124841998285455137338908131 |
| 9 | 0.18398306124833255797187772764544302 |
| 10 | 0.18398306124833929946685349025674957 |
| 11 | 0.18398306124833918404985469216386875 |
| 12 | 0.18398306124833918700689278881066430 |
| 13 | 0.18398306124833918693967757277042711 |
| 14 | 0.18398306124833918694121655021916395 |
| 15 | 0.18398306124833918694118046846226018 |
| 16 | 0.18398306124833918694118129222351397 |
| 17 | 0.18398306124833918694118127301338345 |
| 18 | 0.18398306124833918694118127345475071 |
| 19 | 0.18398306124833918694118127344451095 |
| 20 | 0.18398306124833918694118127344474707 |

TABLE 1. Successive approximations to $\dim(\Lambda_{\pi/6})$

Therefore (see Lemma 4 in Appendix B) we define

$$\gamma = \left(\max_{0\le j\le 2}\gamma_j\right)^{1/3} = 0.68515\ldots \text{ and } K = \left(\max_{0\le j\le 2}K_j\right)\gamma^{-2} = 1.17105\ldots$$

The estimate (6.12) gives us, for $s \geq 0$,

$$\left| \sum_{N=21}^{\infty} d_N(s) \right| \leq \sum_{N=21}^{\infty} \frac{N^{N/2} K^N \gamma^{N(N-1)/2}}{(1-\gamma)\dots(1-\gamma^n)} < 1.3 \times 10^{-18}.$$

Since $\Delta_{20}(s) = 1 + \sum_{N=1}^{20} d_N(s)$, then for all $0 \leq s \leq 1$ we have

$$|\det(I - \mathcal{L}_s) - \Delta_{20}(s)| < 1.3 \times 10^{-18}. \tag{7.1}$$

Let $s_\infty = \dim(\Lambda_{\pi/6})$ denote the largest zero of $\det(I - \mathcal{L}_s)$, and $s_{20}$ the largest zero of $\Delta_{20}$.

If we can find values $s^- < s_{20} < s^+$ with $\Delta_{20}(s^+) \geq 1.3 \times 10^{-18}$ and $\Delta_{20}(s^-) \leq -1.3 \times 10^{-18}$ then by (7.1) we see that $\det(I - \mathcal{L}_{s^+}) > 0$ and $\det(I - \mathcal{L}_{s^-}) < 0$. It then follows by the intermediate value theorem applied to $\det(I - \mathcal{L}_s)$ that $s^- < s_\infty < s^+$. It turns out that the choices $s^- = 0.183983061248339186$ and $s^+ = 0.183983061248339188$ (which agree to 17 decimal places) give $\Delta_{20}(s^+) > 8 \times 10^{-18}$ and $\Delta_{20}(s^-) < -7 \times 10^{-18}$.

We have therefore *rigorously* proved that

$$\dim(\Lambda_{\pi/6}) = 0.183983061248339187 \pm 10^{-18}.$$

Of course this rigorous estimate is more conservative than the empirical estimate we would like to infer from the list of leading zeros $s_N$ of $\Delta_N$ given in Table 1. Here the quality of convergence strongly suggests that

$$\dim(\Lambda_{\pi/6}) \approx 0.183983061248339186941181273444\dots,$$

an accuracy of 30 decimal places.

Indeed for this example we can obtain superior empirical estimates by considering the leading zeros $s_{2r}^{(2)}$ of the functions $\Delta_{2r}^{(2)}$ (i.e. approximations to $\det(I - \mathcal{L}_s^2)$, cf. §3.6). Again working with points up to period 20 we obtain

$$s_{20}^{(2)} = 0.18398306124833918694118127344474173288\dots$$

which is empirically correct to the 38 decimal places given.

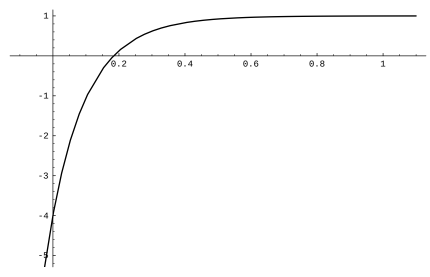


FIGURE 8 . Graph of $s \mapsto \det(I - \mathcal{L}_s)$ for $\Lambda_{\pi/6}$

*Remarks.* Figure 8 shows part of the graph of $\Delta(s) = \det(I - \mathcal{L}_s)$ corresponding to the limit set $\Lambda_{\pi/6}$. Although the graph is concave in the region shown, it is not concave over the whole real line.

For general systems, $\Delta(s)$ is not necessarily locally concave at $s = \dim(\Lambda)$. For example the iterated function scheme given by the contractions $\frac{x}{2}$ and $\frac{x}{2} + \frac{1}{2}$, with limit set the unit interval (whose Hausdorff dimension is 1), has $\Delta(s) = \prod_{r=0}^{\infty}(1 - 2^{1-r-s})$. Note that the zeros of $\Delta$ are those integers not greater than 1. Moreover, it is easy to show that the second derivative $\Delta''(1) > 0$, so that $\Delta$ is locally convex (rather than concave) at $s = 1$.

## 8. Numerical estimates on the dimension of Julia sets for quadratic polynomials

In this section we consider dimension estimates for Julia sets $\mathcal{J}_c$ of quadratic maps $f_c(z) = z^2 + c$.

For practical purposes, our algorithm is most effective in computing the dimension of $\dim(\mathcal{J}_c)$ for $c$ either in the main cardioid of the Mandelbrot set, or $c$ outside of $\mathcal{M}$. In the latter case all periodic points are repelling, while in the former case all periodic points are repelling except for a single attractive fixed point.

First we consider the purely imaginary value $c = i/4$, which lies in the main cardioid of the Mandelbrot set. Table 2 illustrates the successive approximations $s_N$ to $\dim(\mathcal{J}_{i/4})$ arising from our algorithm.

If we take the parameter value $c = -\frac{3}{2} + \frac{2}{3}i$, which lies outside the Mandelbrot set, then the sequence of approximations to the dimension of $\mathcal{J}_c$ are given in Table 3.

For real values of $c$ which are strictly less than $-2$, the Julia set $\mathcal{J}_c$ is a Cantor set completely contained in the real line. For such cases we have, by Corollary 3.1, the faster $O(\delta^{N^2})$ convergence rate to $\dim(\mathcal{J}_c)$, as illustrated in Table 4 for the case $c = -5$.

| $N$ | $N^{th}$ approximation to $\dim(\mathcal{J}_{-5})$ |
|---|---|
| 1 | 0.451399358476417460967595910124138334 9 |
| 2 | 0.4841518684194122992464635900326070715 |
| 3 | 0.4847979587486975778612282908975662571 |
| 4 | 0.4847982943561895699730717563576367090 |
| 5 | 0.4847982944381635057518511943420942957 |
| 6 | 0.4847982944381604305347487891271825909 |
| 7 | 0.4847982944381604305383984765793729512 |
| 8 | 0.4847982944381604305383984781726830747 |

TABLE 4. Successive approximations to $\dim(\mathcal{J}_{-5})$

| $N$ | $N^{th}$ approximation to $\dim(\mathcal{J}_{i/4})$ |
|---|---|
| 3 | 1.1677078534172827136 |
| 4 | 0.9974580934808979848 |
| 5 | 1.0169164188641603339 |
| 6 | 1.0218764720532313644 |
| 7 | 1.0230776911089017648 |
| 8 | 1.0232246810534996595 |
| 9 | 1.0232072525392922127 |
| 10 | 1.0231992637099065199 |
| 11 | 1.0231993120941968028 |
| 12 | 1.0231992857944621198 |
| 13 | 1.0231992888227184780 |
| 14 | 1.0231992890455073830 |
| 15 | 1.0231992890300189633 |
| 16 | 1.0231992890307255210 |
| 17 | 1.0231992890309781268 |
| 18 | 1.0231992890309686742 |
| 19 | 1.0231992890309691466 |
| 20 | 1.0231992890309691251 |

TABLE 2. Successive approximations to $\dim(\mathcal{J}_{i/4})$

For larger negative real values of $c$, the hyperbolicity of $f_c : \mathcal{J}_c \to \mathcal{J}_c$ is more pronounced, so that the constant $0 < \delta < 1$ in the $O(\delta^{N^2})$ estimate is closer to zero, and the convergence to $\dim(\mathcal{J}_c)$ consequently faster. Table 5 illustrates this for $c = -20$ (cf. [BZ, Tableau 2]).

| $N$ | $N^{th}$ approximation to $\dim(\mathcal{J}_{-20})$ |
|---|---|
| 1 | 0.31485651652009699091265279629753355933688857812644665851918 |
| 2 | 0.31850483144363986562810164826944017431378984622904321285835 |
| 3 | 0.31850809576591085725942984004207253452015913804880055477625 |
| 4 | 0.31850809575800523882867786043747732330759968092023152922729 |
| 5 | 0.31850809575800524988789850335472906645586111530021825766595 |
| 6 | 0.31850809575800524988789848098884346788677292871828344714065 |
| 7 | 0.31850809575800524988789848098884348414792438297975066097358 |
| 8 | 0.31850809575800524988789848098884348414792438305840652044425 |

TABLE 5. Successive approximations to $\dim(\mathcal{J}_{-20})$

| $N$ | $N^{th}$ approximation to $\dim(\mathcal{J}_{-3/2+2i/3})$ |
|---|---|
| 1 | 0.7149355610391974853 |
| 2 | 0.9991996994914223217 |
| 3 | 0.8948837401931045135 |
| 4 | 0.8990693400138277172 |
| 5 | 0.9048525377869365908 |
| 6 | 0.9040847144651654898 |
| 7 | 0.9038472818583009063 |
| 8 | 0.9038738383368002502 |
| 9 | 0.9038748469934538668 |
| 10 | 0.9038745896021979531 |
| 11 | 0.9038745956441220338 |
| 12 | 0.9038745968650866636 |
| 13 | 0.9038745968171929578 |
| 14 | 0.9038745968108846487 |
| 15 | 0.9038745968111623979 |
| 16 | 0.9038745968111848616 |

TABLE 3. Successive approximations to $\dim(\mathcal{J}_{-3/2+2i/3})$

*Remarks.*

For $c$ lying outside the Mandelbrot set, $\mathcal{J}_c$ is a Cantor set, and we have the following asymptotic (see [BZ]):

$$\dim(\mathcal{J}_c) \sim \frac{\log 2}{\log 2 + \frac{1}{2}\log|c|} \qquad \text{as } |c| \to \infty.$$

For $c$ inside the main cardioid of the Mandelbrot set the Julia set is a quasi-circle, with $\dim(\mathcal{J}_c)$ strictly greater than one except when $c = 0$ [Sul]. For $|c|$ small we have the following second-order asymptotic due to Ruelle [Ru3],

$$\dim(\mathcal{J}_c) \sim 1 + \frac{|c|^2}{4\log 2}, \qquad \text{as } |c| \to 0.$$

Higher order asymptotics for small $c$ are contained in [WBKS], [CDM], [AMO].

Ruelle [Ru3] proved that, when restricted to either a hyperbolic component of the Mandelbrot set $\mathcal{M}$, or to the exterior, the map $c \mapsto \dim(\mathcal{J}_c)$ is real-analytic. McMullen [McM2] proved that this map is continuous when restricted to the real interval $(c_{\mathrm{Feig}}, 1/4]$ (where $c_{\mathrm{Feig}} \approx -1.401155$ is the Feigenbaum point).

However $c \mapsto \dim(\mathcal{J}_c)$ is discontinuous at various points on the boundary of the Mandelbrot set. The case $c = 1/4$ (for which the Julia set $\mathcal{J}_c$ contains a parabolic fixed point) was studied by Douady, Sentenac & Zinsmeister [DSZ], who proved this is a point of discontinuity (see also [JKP]). Havard & Zinsmeister [HZ] proved that when restricted to the real line, the left derivative of the map $c \mapsto \dim(\mathcal{J}_c)$ at the point $c = 1/4$ is infinite.

For other approaches to numerical computation of Hausdorff dimension of quadratic map Julia sets, see [McM3], [BZ], [WBKS], [Gar].

## 9. Final comments

Our method applies without significant changes to other systems:

(1) Blaschke products: These are maps $f$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ defined by

$$f(z) = \prod_{i=1}^{n} \left( \frac{a_i z + 1}{z + \overline{a}_i} \right)$$

for some finite collection $a_1, \ldots, a_n \in \mathbb{C}$ satisfying $|a_i|^2 = 2$ (i.e. a product of linear fractional transformations each of which preserves the unit disk $\mathbb{D}$). The dynamics of these transformations combine some of the features of both rational maps and Fuchsian groups (see [DM] for more details). The Julia set $\mathcal{J}$ of a Blaschke product is contained in the unit circle, and admits a Markov partition, so in the hyperbolic case we can use Theorem 3 with $d = 1$ to approximate $\dim(\mathcal{J})$ at rate $O(\delta^{N^2})$.

(2) Apollonian packings: A form of Schottky group which does not quite fit into the scheme of the previous section is that of the Apollonian circle packing. This is the limit set $\Lambda$ of a group $\Gamma$ generated by reflections in four mutually tangent circles $C_1, \ldots, C_4$.

There are various estimates on $\dim(\Lambda)$. The best rigorous estimate is $1.300197 < \dim(\Lambda) < 1.314534$, due to Boyd [Boyd]. McMullen [McM3] gives the empirical estimate $\dim(\Lambda) \approx 1.305688$.

Although the Apollonian circle packing is not hyperbolic, it can readily be approximated by hyperbolic systems. For example taking circles $C_{i,\varepsilon}$, $i = 1, \ldots, 4$, with each $C_{i,\varepsilon}$ concentric with $C_i$, and of radius $\varepsilon > 0$ smaller than the radius of $C_i$, we define the reflection group $\Gamma_\varepsilon$ to be generated by reflections in the $C_{i,\varepsilon}$. We can apply our techniques to compute the dimension of the associated limit sets $\Lambda_\varepsilon$, and note that $\lim_{\varepsilon \to 0} \dim(\Lambda_\varepsilon) = \dim(\Lambda)$ by Theorem 1.4 of [McM1].

There are various alternative ways of approximating $\Lambda$. In [MU2], for example, $\Lambda$ is expressed as the limit set of a hyperbolic *infinite* iterated function scheme. Approximating this by some suitably large, yet finite, iterated function scheme, suggests another approach to computing $\dim(\Lambda)$.

## Appendix A: Implementation of the periodic point algorithm

We wish to approximate the Hausdorff dimension of the limit set of a real-analytic conformal iterated function scheme, or the Julia set of a hyperbolic holomorphic Markov map. Here we detail the key steps of the periodic point algorithm, and discuss practical issues concerning its effective implementation.

**Step 1.** Fix a natural number $N$.

**Step 2.** Locate all points of period $\leq N$. There are various ways of doing this, and the method chosen may vary so as to exploit features of the underlying dynamical system.

For the limit set of a Kleinian group, all maps in the associated iterated function scheme are fractional linear. Therefore all compositions are also fractional linear, so that periodic points (i.e. fixed points of suitable compositions) are just roots of quadratic polynomials. Thus the key computational task is to compute the coefficients of relevant compositions (and hence the coefficients of relevant quadratic polynomials). Identifying Möbius maps with elements of $GL(2,\mathbb{C})$ reduces this task to one of matrix multiplication.

In the case of quadratic maps $f_c(z) = z^2 + c$, we use the program [Mul] of M. Muldoon, which locates points of period $n$ by Newton's method. When applied to finding zeros of $f_c^n - \mathrm{id}$, Newton's method is rather unstable, because of its sensitivity to the "initial guess", so it is preferable to use an $n$-dimensional version. That is, we seek vectors $\underline{z} \in \mathbb{C}^n$ which are fixed points of $F(z_1,\ldots,z_n) := (f_c(z_n), f_c(z_1),\ldots,f_c(z_{n-1}))$. Setting $G := I - F$, the Newton iteration then takes the form $\underline{z}^{(i+1)} = \underline{z}^{(i)} - \left[DG(\underline{z}^{(i)})\right]^{-1} G(\underline{z}^{(i)})$. Care is still needed in making the "initial guess" $\underline{z}^{(0)}$, so we interpolate complex numbers $0 = c_0, c_1, \ldots, c_r = c$, with each $|c_{i+1} - c_i|$ suitably small. The periodic points for $f_0$ are known explicitly (they are roots of unity), and each period-$n$ orbit for $f_{c_0}$ is an approximate orbit for $f_{c_1}$, so can serve as an initial guess in Newton's method. This is applied recursively, at each step the true period-$n$ orbit for $f_{c_j}$ serving as the initial guess for the corresponding period-$n$ orbit for $f_{c_{j+1}}$, until eventually the relevant period-$n$ orbit for $f_c = f_{c_r}$ is determined.

**Step 3.** Evaluate relevant derivatives at each point of period $n \le N$, and substitute into the trace formula. If $z_{\underline{i}}$ is a fixed point of $\phi_{\underline{i}} = \phi_{i_{n+1}i_n} \circ \ldots \circ \phi_{i_3 i_2} \circ \phi_{i_2 i_1}$ (a composition of contractions defining the iterated function scheme), then this trace formula is (cf. (3.25))

$$\mathrm{tr}(\mathcal{L}_{s,\underline{i}}) = \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

If the dynamical system is given in terms of a hyperbolic holomorphic Markov map $f$ then the formula (cf. proof of Prop. 11) becomes

$$\mathrm{tr}(\mathcal{L}_{\underline{j}}) = |(f^n)'(z_{\underline{j}})|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}((f^n)'(z_{\underline{j}}))}{|(f^n)'(z_{\underline{j}})|^2}\right)^{-1}.$$

**Step 4.** For each $n \le N$, combine the trace evaluations for all period-$n$ points to form the term (cf. (3.30))

$$d_n(s) = \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\ldots+n_m = n}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\underline{i}\in \mathrm{Fix}(n_l)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

**Step 5.** Sum the $d_n(s)$ to form the function $\Delta_N(s) = 1 + \sum_{n=1}^{N} d_n(s)$.

**Step 6.** Locate the largest real zero $s_N$ of $\Delta_N$. This can be performed to arbitrary precision by means of an appropriate numerical method (eg. Newton's method, secant method, gradient descent, etc.). The value $s_N$ is an approximation to the true Hausdorff dimension.

*Practical Considerations.*

A computer is necessary for the effective implementation of this algorithm.[2] We used a standard UNIX workstation, which was effective in computing points of period up to around $N = 20$ for the quadratic map and reflection group examples considered here.

Programs were written in Mathematica$^{\mathrm{TM}}$ by the first author. For the reflection group example they were complemented by C programs of I. Rivin. For the quadratic map example the periodic points were located using M. Muldoon's program [Mul].

Further details are available via the website [JP2].

## Appendix B: Combining contraction ratios

The following lemma, which describes how to combine different contraction ratios, is useful to obtain estimates on the super-exponential rate of convergence, which in turn can be used to prove *rigorous* dimension estimates (see §7). Part (a) of the lemma gives an optimal estimate on this convergence rate. Part (b), while giving a more conservative estimate, is often of more practical use, since the constant $K$ is often much smaller than that in part (a).

**Lemma 4.** *Suppose $K_0, \ldots, K_{p-1} > 0$, and $\gamma_0, \ldots, \gamma_{p-1} \in (0,1)$. Suppose $(\alpha_i)$ is a sequence of positive real numbers, such that*

$$\alpha_{pn+j} \leq K_j \gamma_j^n \quad \text{for all } n \geq 0, \text{ and } j = 0, \ldots, p-1.$$

*Let $\beta_1 \geq \beta_2 \geq \ldots$ be the non-decreasing rearrangement of the sequence $(\alpha_i)$. Then for all $i \geq 0$ we have*

$$\beta_i \leq K\gamma^i,$$

*where we can choose $0 < \gamma < 1$ and $K > 0$ to satisfy either*

*(a)*

$$\gamma = \exp\left(\left(\sum_{j=0}^{p-1} \frac{1}{\log \gamma_j}\right)^{-1}\right),$$

*and*

$$K = \max\left(\gamma^{-1-2p+\sum_{j=0}^{p-1} \frac{\log K_j}{\log \gamma_j}}, \max_{0 \leq j \leq p-1} K_j\right).$$

*or (b)*

$$\gamma = \left(\max_{0 \leq j \leq p-1} \gamma_j\right)^{1/p}$$

[2] If cruder dimension estimates are of use then approximations based on points of very low period (eg. fixed points and period-2 points, say) are sometimes feasible "by hand".

*and*

$$K = \left( \max_{0 \le j \le p-1} K_j \right) \gamma^{1-p}.$$

*Proof.* (a) For $L > 0$, define $M(L)$ to be the number of terms in the sequence $(\alpha_i)$ which are greater than or equal to $L$. We then have $\beta_{M(L)} \ge L > \beta_{M(L)+1}$ for all $L > 0$.

Define the sequence $(\delta_i)$ by $\delta_{pn+j} = K_j \gamma_j^n$ for $n \ge 0$ and $j = 0, \dots, p-1$. Now if $\alpha_{pn+j} \ge L$, then certainly $\delta_{pn+j} = K_j \gamma_j^n \ge L$. Thus if we define $N(L)$ to be the number of terms in the sequence $(\delta_i)$ which are greater than or equal to $L$, then we clearly have $N(L) \ge M(L)$.

Now

$$K_j \gamma_j^n \ge L \quad \text{if and only if} \quad n \le \frac{\log L - \log K_j}{\log \gamma_j},$$

so we have

$$N(L) = \sum_{j=0}^{p-1} \left( \left[ \frac{\log L - \log K_j}{\log \gamma_j} \right] + 1 \right) = p + \sum_{j=0}^{p-1} \left[ \frac{\log L - \log K_j}{\log \gamma_j} \right],$$

where $[\cdot]$ denotes the integer part of a real number.

Let $\epsilon_1 \ge \epsilon_2 \ge \dots$ be the non-increasing rearrangement of the sequence $(\delta_i)$. Suppose we have that

$$\epsilon_i \le K' \gamma^i, \tag{B.1}$$

where $K' > 0$ and $0 < \gamma < 1$ are chosen to be optimal, in the sense that $\limsup_{i \to \infty} \epsilon_i^{1/i} = \gamma$, and $K' = \sup_{i \ge 0} \epsilon_i / \gamma^i$.

Now for all $L > 0$ we have

$$\epsilon_{N(L)} \ge L > \epsilon_{N(L)+1}, \tag{B.2}$$

so taking $N(L)^{th}$ roots in this inequality, then taking limit suprema as $L \to 0$ (so that $N(L) \to \infty$), we obtain

$$\gamma = \limsup_{L \to 0} L^{1/N(L)} = \limsup_{L \to 0} L^{\left( p + \log L \sum \frac{1}{\log \gamma_j} - \sum \frac{\log K_j}{\log \gamma_j} \right)^{-1}}.$$

Taking logarithms gives

$$\log \gamma = \limsup_{L \to 0} \frac{\log L}{p + \log L \sum \frac{1}{\log \gamma_j} - \sum \frac{\log K_j}{\log \gamma_j}} = \frac{1}{\sum_{j=0}^{p-1} \frac{1}{\log \gamma_j}},$$

so that

$$\gamma = \exp \left( \left( \sum_{j=0}^{p-1} \frac{1}{\log \gamma_j} \right)^{-1} \right).$$

Now for any positive integer $i$ which is of the form $i = N(L) + 1$, for some $L > 0$, the right hand side of (B.2) gives us

$$\frac{\epsilon_i}{\gamma^i} < \frac{L}{\gamma^{N(L)+1}}. \qquad (B.3)$$

If the integer $i \geq 0$ is *not* of the form $N(L) + 1$, then there must exist $q \geq 2$ (note also that $q \leq p$) consecutive terms in the re-arranged sequence $(\epsilon_i)$ which are all equal to some $L > 0$. That is, we have

$$\epsilon_{k+1} = \ldots = \epsilon_{i-1} = \ldots = \epsilon_{k+q} = L > \epsilon_{k+q+1}.$$

Note that $k + q = N(L)$. Since $0 < \gamma < 1$ then

$$\epsilon_i/\gamma^i < \epsilon_{k+q}/\gamma^{k+q} = L/\gamma^{N(L)} < L/\gamma^{N(L)+1}. \qquad (B.4)$$

Combining (B.3), (B.4) we see that

$$\sup_{i\geq 1} \epsilon_i/\gamma^i \leq \sup_{L>0} L/\gamma^{N(L)+1}. \qquad (B.5)$$

Estimating the denominator in (B.5) we have

$$\begin{aligned}
\gamma^{N(L)+1} &= \gamma^{1+p+\sum_{j=0}^{p-1}\left(\left[\frac{\log L - \log K_j}{\log \gamma_j}\right]+1\right)} \\
&\geq \gamma^{1+2p+\sum \frac{\log L - \log K_j}{\log \gamma_j}} \\
&= \exp\left(\frac{1+2p}{\sum \frac{1}{\log \gamma_j}}\right) \exp(\log L) \exp\left(\frac{-\sum \frac{\log K_j}{\log \gamma_j}}{\sum \frac{1}{\log \gamma_j}}\right) \\
&= \gamma^{1+2p} L \gamma^{-\sum \frac{\log K_j}{\log \gamma_j}}.
\end{aligned}$$

Therefore we have

$$\sup_{L>0} L/\gamma^{N(L)+1} \leq \gamma^{-1-2p+\sum_{j=0}^{p-1} \frac{\log K_j}{\log \gamma_j}}. \qquad (B.6)$$

So from (B.5) we obtain

$$\sup_{i\geq 1} \epsilon_i/\gamma^i \leq \gamma^{-1-2p+\sum_{j=0}^{p-1} \frac{\log K_j}{\log \gamma_j}}. \qquad (B.7)$$

Now $K' = \sup_{i\geq 0} \epsilon_i/\gamma^i$, and $\epsilon_0/\gamma^0 = \epsilon_0 = \max_{0\leq j\leq p-1} K_j$. So combining this with (B.7) gives us

$$K' = \sup_{i\geq 0} \epsilon_i/\gamma^i \leq \max\left(\gamma^{-1-2p+\sum_{j=0}^{p-1} \frac{\log K_j}{\log \gamma_j}}, \max_{0\leq j\leq p-1} K_j\right). \qquad (B.8)$$

Now defining $K$ to be the right-hand side of (B.8) we use (B.1) to see that

$$\epsilon_i \leq K'\gamma^i \leq K\gamma^i.$$

Moreover, since each $\alpha_i \le \delta_i$ for all $i \ge 0$, then rearranging both sequences gives $\beta_i \le \epsilon_i$ for all $i \ge 0$, from which we deduce that

$$\beta_i \le \epsilon_i \le K\gamma^i,$$

as required.

(b) Define the sequence $(\delta_i)_{i=0}^\infty$ as in the proof of part (a), and define $(\tau_i)_{i=0}^\infty$ by $\tau_{pn+k} = (\max_{0\le j\le p-1} K_j)(\max_{0\le j\le p-1}\gamma_j)^n$ for $n \ge 0$ and $k = 0, \ldots, p-1$. Then we have $\alpha_i \le \delta_i \le \tau_i$ for all $i \ge 0$, and moreover

$$\begin{aligned}\tau_i &\le (\max_{0\le j\le p-1} K_j)(\max_{0\le j\le p-1}\gamma_j)^{\frac{i-(p-1)}{p}} \\ &= (\max_{0\le j\le p-1} K_j)\gamma^{1-p}\gamma^i.\end{aligned}$$

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