On the Ruelle eigenvalue sequence

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Abstract. For certain real analytic data, we show that the eigenvalue sequence of the associated transfer operator $\mathcal{L}$ is insensitive to the holomorphic function space on which $\mathcal{L}$ acts. Explicit bounds on this eigenvalue sequence are established.

1. *Introduction*

For compact $X \subset \mathbb{C}^d$, and appropriate real analytic $T_i : X \to X$ and $w_i : X \to \mathbb{C}$, Ruelle [**Rue1**] considered the action of the transfer operator $\mathcal{L}f := \sum_i w_i \cdot f \circ T_i$ on $U(D)$, where $D \subset \mathbb{C}^d$ is a domain on which all the T_i's and w_i's are holomorphic, and $U(D)$ consists of those holomorphic functions on D which extend continuously to the closure of D. Ruelle proved that $\mathcal{L} : U(D) \to U(D)$ is nuclear, hence in particular compact, and that its eigenvalue sequence $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$, henceforth referred to as the *Ruelle eigenvalue sequence*, is given by the reciprocals of the zeros of a dynamical determinant Δ (see (9) for the definition).

In view of its various interpretations and applications (e.g. correlation decay rates [**Bal, CPR**], Fourier resonances [**Rue2**], Laplacians for hyperbolic surfaces [**Pol1, PR**], Feigenbaum period-doubling [**AAC, CCR, JMS, Pol2**]), it is desirable to establish explicit bounds on the Ruelle eigenvalue sequence. In the case where D may be chosen as a ball, and the T_i all map D within the concentric ball whose radius is $r < 1$ times that of D, we establish (Theorem 3.2) the stretched-exponential bound

$$|\lambda_n(\mathcal{L})| < \frac{W}{r^d} n^{1/2}\, r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \text{for all } n \geq 1\,, \tag{1}$$

where $W := \sup_{z\in D} \sum_i |w_i(z)|$.

We go on to investigate properties of transfer operators acting on other spaces of holomorphic functions, and prove (Theorem 4.2) that the Ruelle eigenvalue sequence is in a sense *universal*: for a wide range of domains D, and a broad class of spaces $A(D)$ of holomorphic functions on D, the eigenvalue sequence of $\mathcal{L} : A(D) \to A(D)$ is precisely the Ruelle eigenvalue sequence. This universality suggests the possibility of sharpening the estimate (1), by adapting the proof of Theorem 3.2 to some other space $A(D)$. In particular, the choice of $A(D)$ as the Hardy space $H^2(D)$ is known to yield a concrete eigenvalue bound for $\mathcal{L} : H^2(D) \to H^2(D)$ (see [**BJ**]). Intriguingly, this bound turns out to be *complementary* to (1): in every dimension d, and for every $r < 1$, (1) is superior for sufficiently small n, while the Hardy space bound is superior for sufficiently large n. If $N(r, d)$ denotes the integer such that (1) gives the sharper bound on $|\lambda_n(\mathcal{L})|$ precisely for $1 \le n \le N(r, d)$, then both $r \mapsto N(r, d)$ and $d \mapsto N(r, d)$ are increasing (cf. Corollary 4.4, Remark 4.5); in other words, (1) is more useful if the T_i are weakly contracting, or if the ambient dimension is high.

2. *Transfer operators on favourable spaces of holomorphic functions*

NOTATION 2.1. Let $\mathbb{N}$ denote the set of strictly positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, equip $\mathbb{C}^d$ with the Euclidean inner product $(\cdot, \cdot)_{\mathbb{C}^d}$, the corresponding norm $\| \cdot \|_{\mathbb{C}^d}$, and the induced Euclidean metric, denoted δ. For $X \subset \mathbb{C}^d$ we use $\Delta_\varepsilon(X) = \left\{ z \in \mathbb{C}^d \,|\, \delta(z, X) < \varepsilon \right\}$ for the Euclidean ε-neighbourhood of X. The set of all bounded domains (non-empty connected open subsets) in $\mathbb{C}^d$ will be denoted by $\mathcal{D}_d$. For two bounded open sets $\Delta_1, \Delta_2 \subset \mathbb{C}^d$ we write $\Delta_1 \subset\subset \Delta_2$ to mean that $\overline{\Delta_1} \subset \Delta_2$.

Let $B = (B, \|\cdot\|_B)$ be a Banach space. We often write $\|\cdot\|$ instead of $\|\cdot\|_B$ whenever this does not lead to confusion. For $X \subset \mathbb{C}^d$ compact and $D \in \mathcal{D}_d$ define

$$
\begin{aligned}
Hol(D, B) &:= \{ f : D \to B \,|\, f \text{ holomorphic} \} \\
C(X, B) &:= \{ f : X \to B \,|\, f \text{ continuous} \}, \ \|f\|_{C(X,B)} := \sup_{x \in X} \|f(x)\|_B \\
U(D, B) &:= \left\{ f : \overline{D} \to B \,|\, f \in C(\overline{D}, B) \cap Hol(D, B) \right\}, \ \|f\|_{U(D,B)} := \sup_{z \in \overline{D}} \|f(z)\|_B \, .
\end{aligned}
$$

Note that $C(X, B)$ and $U(D, B)$ are Banach spaces when equipped with the indicated norms, while $Hol(D, B)$ is a Fréchet space when equipped with the topology of uniform convergence on compact subsets of D. If $(B, \| \cdot \|) = (\mathbb{C}, | \cdot |)$ then we use $C(X)$, $Hol(D)$, and $U(D)$ to denote $C(X, \mathbb{C})$, $Hol(D, \mathbb{C})$, and $U(D, \mathbb{C})$ respectively.

We use $L(B)$ to denote the space of bounded linear operators from a Banach space $(B, \| \cdot \|)$ to itself, always equipped with the induced operator norm.

If T is holomorphic on some $D \in \mathcal{D}_d$, its derivative at $z \in D$ is denoted by $T'(z)$.

DEFINITION 2.2. Let $\mathcal{I}$ be a non-empty countable set. For $D \in \mathcal{D}_d$, a collection $(T_i)_{i \in \mathcal{I}} = (T_i, D)_{i \in \mathcal{I}}$ of holomorphic maps $T_i \in U(D, \mathbb{C}^d)$ is called a *holomorphic map system* (on D) if $\cup_{i \in \mathcal{I}} T_i(D) \subset\subset D$.

Write $T_{\underline{i}} := T_{i_n} \circ \cdots \circ T_{i_1}$ for $\underline{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$, $n \in \mathbb{N}$.

For $X \subset \mathbb{C}^d$ compact, a collection $(T_i)_{i \in \mathcal{I}} = (T_i, X)_{i \in \mathcal{I}}$ of maps $T_i : X \to X$ is a C^ω *map system* (on X) if there exists $D \in \mathcal{D}_d$ with $X \subset\subset D$ such that each T_i extends holomorphically to D and $(T_i, D)_{i \in \mathcal{I}}$ is a holomorphic map system. Any such D is called *admissible* for the C^ω map system $(T_i, X)_{i \in \mathcal{I}}$.

For $n \in \mathbb{N}$, a C^ω map system $(T_i, X)_{i \in \mathcal{I}}$ is called *complex n-contracting* (or simply *complex contracting*) if there exists $D \in \mathcal{D}_d$ with $X \subset\subset D$, such that $T'_{\underline{i}} \in U(D, L(\mathbb{C}^d))$

for every $\underline{i} \in \mathcal{I}^n$ and

$$\sup_{\underline{i}\in\mathcal{I}^n} \left\| T'_{\underline{i}} \right\|_{U(D,L(\mathbb{C}^d))} < 1 \,. \tag{2}$$

Note that if $\mathcal{I}$ is finite then (2) is implied by the condition $\sup_{\underline{i}\in\mathcal{I}^n} \|T'_{\underline{i}}\|_{C(X,L(\mathbb{C}^d))} < 1$.

EXAMPLE 2.3. If $X = [0,1] \subset \mathbb{C}$, define the *Gauss map system* $(T_i)_{i\in\mathbb{N}}$ by $T_i(x) = 1/(i+x)$ (the T_i are the inverse branches to the Gauss map $x \mapsto 1/x \pmod 1$ on X). This is a C^ω map system on X: for example if $D \subset \mathbb{C}$ is the open disc of radius $3/2$ centred at the point 1 then $(T_i, D)_{i\in\mathcal{I}}$ is a holomorphic map system. The system is also complex contracting, because $\sup_{\underline{i}\in\mathcal{I}^2} \|T'_{\underline{i}}\|_{U(D)} = |T'_{(1,1)}(-1/2)| = 4/9 < 1$ (note we cannot choose $n = 1$ in (2), because $T'_1(0) = -1$).

Complex contraction guarantees the existence of an admissible domain, and this domain may be chosen arbitrarily close to X:

LEMMA 2.4. *If a C^ω map system on X is complex contracting then there exists a family $\{D_\theta\}_{\theta\in(0,\Theta)}$ of admissible domains, such that $\cap_{\theta\in(0,\Theta)} D_\theta = X$.*

Proof. Let $(T_i)_{i\in\mathcal{I}}$ denote the C^ω map system on X. Choose $n \in \mathbb{N}$ and $D \in \mathcal{D}_d$ such that $\gamma := \sup_{\underline{i}\in\mathcal{I}^n} \|T'_{\underline{i}}\|_{U(D,L(\mathbb{C}^d))} < 1$. From the several variables mean value theorem [**Ave**, Thm. 2.3], for each $\underline{i} \in \mathcal{I}^n$, the map $T_{\underline{i}}$ is γ-Lipschitz, with respect to Euclidean distance δ, on any convex subset of D. Now set $\beta := \gamma^{1/n} < 1$, and define the distance

$$\operatorname{dist}(x,y) = \sup_{\underline{i}\in\mathcal{I}^{n-1}} \sum_{k=0}^{n-1} \beta^{n-1-k} \delta(T_{P_k(\underline{i})}(x), T_{P_k(\underline{i})}(y)) \,,$$

where for $1 \le k \le n-1$, $P_k : \mathcal{I}^{n-1} \to \mathcal{I}^k$ denotes the projection $P_k\underline{i} = (i_1, \dots, i_k)$ onto the first k coordinates, with the convention that $T_{P_0\underline{i}} = \mathrm{id}$. Note that for each $i \in \mathcal{I}$, the map T_i is β-Lipschitz, with respect to dist, on any convex subset of D. Moreover, dist and δ generate the same topology on δ-compact subsets of D. To see this observe that on the one hand we clearly have $\delta(x,y) \le \beta^{1-n}\operatorname{dist}(x,y)$ for every $x, y \in D$. On the other hand, if K is a δ-compact convex subset of D, then by Cauchy's theorem there is $C > 0$ such that $\|T'_{\underline{i}}\|_{C(K,L(\mathbb{C}^d))} \le C$ for every $\underline{i} \in \mathcal{I}^k$, $1 \le k \le n-1$. Thus by the mean value theorem $\operatorname{dist}(x,y) \le C \sum_{k=0}^{n-1} \beta^{n-1-k}\delta(x,y)$ for every $x, y \in K$.

Since X is compactly contained in the domain D, there exists $\varepsilon > 0$ such that the Euclidean neighbourhood $\Delta_\varepsilon(X)$ is contained in D. Setting $\Theta := \varepsilon\beta^{n-1}$, we see that $D_\theta := \left\{ z \in \mathbb{C}^d \,|\, \operatorname{dist}(z,X) < \theta \right\} \subset \Delta_\varepsilon(X)$ for all $\theta \in (0,\Theta]$, and that $\cap_{\theta\in(0,\Theta]} D_\theta = X$. If $z \in D_\theta$, and $x \in X$ satisfies $\operatorname{dist}(z,X) = \operatorname{dist}(z,x)$, then $x, z \in \Delta_\varepsilon(x)$, a convex subset of D, so $\operatorname{dist}(T_i(z), T_i(x)) \le \beta \operatorname{dist}(z,x)$. Therefore $\operatorname{dist}(T_i(z), X) \le \operatorname{dist}(T_i(z), T_i(x)) \le \beta\operatorname{dist}(z,x) = \beta\operatorname{dist}(z,X)$, and hence $\cup_{i\in\mathcal{I}} T_i(D_\theta) \subset D_{\beta\theta} \subset\subset D_\theta$, so D_θ is admissible for $(T_i, X)_{i\in\mathcal{I}}$. □

REMARK 2.5. The above proof shows that if a C^ω map system on X is complex 1-contracting then all sufficiently small Euclidean ε-neighbourhoods $\Delta_\varepsilon(X)$ are admissible. This is not the case for the Gauss map system $T_i(z) = 1/(z+i)$ on $X = [0,1] \subset \mathbb{C}$: no Euclidean ε-neighbourhood is admissible, since $\delta(T_1(-\varepsilon), X) > \varepsilon$.

DEFINITION 2.6. Let $\mathcal{I}$ be a non-empty countable set. A *holomorphic weight system* on $D \in \mathcal{D}_d$ is a collection $(w_i)_{i\in\mathcal{I}} = (w_i, D)_{i\in\mathcal{I}}$ of holomorphic functions (called *weight functions*) $w_i \in U(D)$ such that $\sum_{i\in\mathcal{I}} \|w_i\|_{U(D)} < \infty$.

For $X \subset \mathbb{C}^d$ compact, a collection $(w_i)_{i\in\mathcal{I}} = (w_i, X)_{i\in\mathcal{I}}$ of maps $w_i : X \to \mathbb{C}$ is a C^ω *weight system* (on X) if there exists $D \in \mathcal{D}_d$ with $X \Subset D$ such that $(w_i, D)_{i\in\mathcal{I}}$ is a holomorphic weight system. Any such D is called *admissible* for $(w_i, X)_{i\in\mathcal{I}}$.

If $(T_i)_{i\in\mathcal{I}}$ is a holomorphic (respectively, C^ω) map system and $(w_i)_{i\in\mathcal{I}}$ is a holomorphic (respectively, C^ω) weight system then $(T_i, w_i)_{i\in\mathcal{I}}$ is called a *holomorphic (respectively, C^ω) map-weight system*. A domain $D \in \mathcal{D}_d$ is called *admissible* for a C^ω map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$ if it is admissible for both $(T_i, X)_{i\in\mathcal{I}}$ and $(w_i, X)_{i\in\mathcal{I}}$.

With each holomorphic map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$ we associate a linear operator,

$$\mathcal{L}f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i\,, \tag{3}$$

called the *transfer operator*. It will be seen that the transfer operator $\mathcal{L}$ preserves, and acts compactly upon, the following class of spaces of holomorphic functions.

DEFINITION 2.7. For $D \in \mathcal{D}_d$, a Banach space $A = A(D)$ of functions $f : D \to \mathbb{C}$ holomorphic on D is called a *favourable space of holomorphic functions (on D)* if

(i) for each $z \in D$, the point evaluation functional $f \mapsto f(z)$ is continuous on A, and

(ii) A contains $U(D)$, and the natural embedding† $U(D) \hookrightarrow A$ has norm 1.

REMARK 2.8. Let $D \in \mathcal{D}_d$. Then $U(D)$ is trivially a favourable space of holomorphic functions. Other examples include, for $p \in [1,\infty]$, *Bergman spaces* $L^p_{Hol}(D)$ (see [**Ran**, Ch. I, Cor. 1.7, 1.10]) and *Hardy spaces* $H^p(D)$ (see [**Kra**, Ch. 8.3]). If $p = 2$ and D has C^2 boundary, then $H^2(D)$ can be identified with the $L^2(\partial D, \sigma)$-closure of $U(D)$, where σ denotes $(2d-1)$-dimensional Lebesgue on the boundary ∂D, normalised so that $\sigma(\partial D) = 1$. In particular, $H^2(D)$ is a Hilbert space with inner product given by $(f, g) = \int_{\partial D} f^*\,\overline{g^*}\,d\sigma$, where, for $h \in H^2(D)$, the symbol h^* denotes the corresponding nontangential limit function in $L^2(\partial D, \sigma)$ — see [**Kra**, Ch. 1.5 and 8].

Recall (see e.g. [**Pie**, 1.7.1]) that a linear operator $L : B \to B$ on a Banach space† B is *p-nuclear* if there exist sequences $b_i \in B$ and $l_i \in B^*$ (the strong dual of B) with $\sum_i(\|b_i\|\,\|l_i\|)^p < \infty$, such that $L(b) = \sum_{i=1}^\infty l_i(b)b_i$ for all $b \in B$. The operator is *strongly nuclear* (or *nuclear of order zero*) if it is p-nuclear for every $p > 0$. It turns out that certain natural embeddings between favourable spaces are strongly nuclear:

LEMMA 2.9. *Let D and D' be domains in $\mathbb{C}^d$ such that $D' \Subset D$. Let A and A' be favourable Banach spaces of holomorphic functions on D and D' respectively. Then $A \subset A'$, and the natural embedding $J : A \hookrightarrow A'$, defined by $Jf = f|_{D'}$, is strongly nuclear.*

† The embedding $U(D) \hookrightarrow A$ is automatically continuous: continuity of point evaluation on both A and $U(D)$ implies that it has closed graph; cf. the proof of Lemma 2.9.
† See [**Gro**, II, Déf. 1, p. 3] for the generalisation to locally convex spaces.

Proof. Choose $D'' \in \mathcal{D}_d$ with $D' \subset\subset D'' \subset\subset D$, and consider the natural embeddings

$$A \overset{J_1}{\hookrightarrow} Hol(D'') \overset{J_2}{\hookrightarrow} U(D') \overset{J_3}{\hookrightarrow} A' .$$

Clearly $J = J_3 J_2 J_1$. The unit ball of $U(D')$ is a neighbourhood in $Hol(D'')$, so the map J_2 is bounded. But the Fréchet space $Hol(D'')$ is nuclear [**Gro**, II, Cor., p. 56], so J_2 is p-nuclear for every $p > 0$ by [**Gro**, II, Cor. 4, p. 39, Cor. 2, p. 61]. It thus suffices to show that J_1 and J_3 are continuous by [**Gro**, I, p. 84, II, p. 9].

Now, J_3 is continuous since A' is favourable. Finally, to see that J_1 is continuous we note that, by the closed graph theorem (see e.g. [**Scha**, Ch. III, 2.3]), it is enough to show that if $f_n \to f$ in A, and $J_1 f_n \to g$ in $Hol(D'')$, then $g = J_1 f = f|_{D''}$. Since point evaluation is continuous on A, $f_n(z) \to f(z)$ for all $z \in D$ and in particular for all $z \in D''$. But point evaluation is also continuous on $Hol(D'')$, so $f_n(z) = J_1 f_n(z) \to g(z)$ as $n \to \infty$ for all $z \in D''$. Therefore $g = f|_{D''}$. □

Favourable spaces A are always invariant under the transfer operator $\mathcal{L}$, and the restricted operator (henceforth denoted by $\mathcal{L}_A$) is always compact, indeed strongly nuclear:

PROPOSITION 2.10. *Let $(T_i, w_i, D)_{i\in\mathcal{I}}$ be a holomorphic map-weight system. The corresponding transfer operator leaves invariant every favourable space A of holomorphic functions on D, and $\mathcal{L}_A : A \to A$ is strongly nuclear.*

Proof. Choose $D' \in \mathcal{D}_d$ with $\cup_{i\in\mathcal{I}} T_i(D) \subset\subset D' \subset\subset D$. First we observe that $\hat{\mathcal{L}} f := \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : U(D') \to U(D)$. To see this, fix $f \in U(D')$ and note that $w_i \cdot f \circ T_i \in U(D)$ with $\|w_i \cdot f \circ T_i\|_{U(D)} \leq \|w_i\|_{U(D)} \|f\|_{U(D')}$ for every $i \in \mathcal{I}$. But since $\|\hat{\mathcal{L}} f\|_{U(D)} \leq \sum_{i\in\mathcal{I}} \|w_i\|_{U(D)} \|f\|_{U(D')}$ and $\sum_{i\in\mathcal{I}} \|w_i\|_{U(D)} < \infty$ by hypothesis, we conclude that $\hat{\mathcal{L}} f \in U(D)$ and that $\hat{\mathcal{L}}$ is continuous.

Since A is favourable, the embedding $\hat{J} : U(D) \hookrightarrow A$ is continuous, and the embedding $J : A \hookrightarrow U(D')$ is p-nuclear for every $p > 0$ by Lemma 2.9. Moreover, if $f \in A$ then $\mathcal{L} f = \hat{J}\hat{\mathcal{L}} J f \in A$. Thus A is $\mathcal{L}$-invariant, and the operator $\mathcal{L}_A = \hat{J}\hat{\mathcal{L}} J$ is p-nuclear for any $p > 0$. □

REMARK 2.11. Strong nuclearity of the transfer operator on spaces of holomorphic functions is not new (the original result of this kind is [**Rue1**], but see also e.g. [**GLZ, JP, May3**]); the novelty of Proposition 2.10 is in the breadth of spaces covered.‡

3. *Eigenvalue bounds*

For favourable A, the compactness of $\mathcal{L}_A$ means its spectrum consists of a countable set of eigenvalues, each with finite algebraic multiplicity, together with a possible accumulation point at 0. We wish to obtain bounds on the *eigenvalue sequence* $\lambda(\mathcal{L}_A) := \{\lambda_n(\mathcal{L}_A)\}_{n=1}^{\infty}$, i.e. the sequence of all eigenvalues of $\mathcal{L}_A$ counting algebraic multiplicities and ordered by decreasing modulus.†

‡ Actually the result can be further extended to certain locally convex spaces of holomorphic functions, including $Hol(D)$.

† By convention distinct eigenvalues with the same modulus can be written in any order (see e.g. [**Pie**, 3.2.20]).

If $L : B_1 \to B_2$ is a continuous operator between Banach spaces then for $k \ge 1$, its k*-th approximation number* $a_k(L)$ is defined as

$$a_k(L) = \inf \{\|L - K\| \mid K : B_1 \to B_2 \text{ linear with } \operatorname{rank}(K) < k\} \ .$$

PROPOSITION 3.1. *For a C^ω map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$ such that $(T_i)_{i\in\mathcal{I}}$ is complex contracting, and a favourable space $A = A(D)$ such that $D \in \mathcal{D}_d$ is admissible,*

$$|\lambda_n(\mathcal{L}_A)| \le W n^{1/2} \prod_{k=1}^{n} a_k(J)^{1/n} \quad \textit{for all } n \ge 1\,, \tag{4}$$

where $W := \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)|$, $D' \in \mathcal{D}_d$ is such that $\cup_{i\in\mathcal{I}} T_i(D) \subset D' \Subset D$, and $J : A(D) \hookrightarrow U(D')$ is the canonical embedding.

Proof. Since $A(D)$ is favourable, the embedding $\hat{J} : U(D) \hookrightarrow A(D)$ is continuous of norm 1. Next observe that $\hat{\mathcal{L}} f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : U(D') \to U(D)$ (see the proof of Proposition 2.10) with $\|\hat{\mathcal{L}}\| \le W$. To see the latter note that for $f \in U(D')$ we have $|f(T_i(z))| \le \|f\|_{U(D')}$ for every $z \in D$, $i \in \mathcal{I}$; thus by the maximum principle $\|\hat{\mathcal{L}} f\|_{U(D)} = \sup_{z\in D} |(\hat{\mathcal{L}} f)(z)| \le \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)|\ |f(T_i(z))| \le W \|f\|_{U(D')}$.

Now clearly $\mathcal{L}_A = \hat{J}\hat{\mathcal{L}}J$, so

$$a_k(\mathcal{L}_A) \le \|\hat{J}\hat{\mathcal{L}}\| a_k(J) \le W a_k(J) \quad \text{for all } k \ge 1\,, \tag{5}$$

since in general $a_k(L_1 L_2) \le \|L_1\|\ a_k(L_2)$ whenever L_1 and L_2 are bounded operators between Banach spaces (see [**Pie**, 2.2]). Moreover, since $\mathcal{L}_A$ is compact, Weyl's inequality (see e.g. [**Hin**]) asserts that $\prod_{k=1}^n |\lambda_k(\mathcal{L}_A)| \le n^{n/2} \prod_{k=1}^n a_k(\mathcal{L}_A)$ for every $n \in \mathbb{N}$.‡

Together with (5) this yields (4), because $|\lambda_n(\mathcal{L}_A)| \le \prod_{k=1}^n |\lambda_k(\mathcal{L}_A)|^{1/n}$. □

Taking $A(D) = U(D)$, the *Ruelle eigenvalue sequence* $\lambda(\mathcal{L}_{U(D)})$ can be bounded as follows:

THEOREM 3.2. *Suppose the Euclidean ball $D \subset \mathbb{C}^d$ is an admissible domain for a C^ω map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$, and that $\cup_{i\in\mathcal{I}} T_i(D)$ is contained in the concentric ball whose radius is $r < 1$ times that of D. Setting $W := \sup_{z\in B} \sum_{i\in\mathcal{I}} |w_i(z)|$, the Ruelle eigenvalue sequence $\lambda(\mathcal{L}_{U(D)})$ can be bounded by*

$$|\lambda_n(\mathcal{L}_{U(D)})| < \frac{W}{r^d} n^{1/2}\ r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \textit{for all } n \ge 1\,. \tag{6}$$

If $d = 1$ then

$$|\lambda_n(\mathcal{L}_{U(D)})| \le W n^{1/2}\ r^{(n-1)/2} \quad \textit{for all } n \ge 1\,. \tag{7}$$

Proof. Without loss of generality let $D = D_1$ be the open unit ball, and let the smaller concentric ball be D_r, the ball of radius r centred at 0. Let $J : U(D_1) \hookrightarrow U(D_r)$ be the canonical embedding. From [**Far**, Prop. 2.1 (a)] it follows that $a_l(J) \le r^{t_l}$, where $t_l := k$ for $\binom{k-1+d}{d} < l \le \binom{k+d}{d}$, hence $\prod_{l=1}^n a_l(J)^{1/n} \le r^{\frac{1}{n}\sum_{l=1}^n t_l}$. If $d = 1$ then

‡ This is a Banach space version of Weyl's original inequality in Hilbert space (see [**Wey**]). Note that the constant $n^{n/2}$ is optimal (see [**Hin**]).

$\frac{1}{n}\sum_{l=1}^{n} t_l = \frac{1}{n}\sum_{l=1}^{n}(l-1) = (n-1)/2$, and (7) follows from (4). More generally $t_l \geq (d!)^{1/d} l^{1/d} - d$, so that

$$\frac{1}{n}\sum_{l=1}^{n} t_l \geq -d + (d!)^{1/d}\frac{1}{n}\sum_{l=1}^{n} l^{1/d} > -d + (d!)^{1/d}\frac{d}{d+1}n^{1/d}$$

using the estimate $\sum_{l=1}^{n} l^{1/d} > \int_{x=0}^{n} x^{1/d} = \frac{d}{d+1}n^{1+1/d}$, and (6) follows from (4). □

4. *Universality of the Ruelle eigenvalue sequence*

If $(T_i, w_i)_{i\in\mathcal{I}}$ is a C^ω map-weight system with complex contracting $(T_i)_{i\in\mathcal{I}}$ then, in view of Lemma 2.4 and Proposition 2.10, there is some freedom in the choice of an admissible D, and a favourable space $A = A(D)$ on which to consider the transfer operator $\mathcal{L}_A$. The purpose of this section is to show that the eigenvalue sequence of $\mathcal{L}_A$ is in fact independent of A: it is always equal to the Ruelle eigenvalue sequence $\lambda(\mathcal{L}_{U(D)})$ (see Corollary 4.3). For this, we first require some facts from the Fredholm theory originally developed by Grothendieck [**Gro**]. If B is a Banach space, we denote by $N_p(B)$ ($p > 0$) the quasi-Banach operator ideal of p-nuclear operators on B (cf. [**Pie**, D.1.4, 1.7.1]). If $p \leq 2/3$ then $N_p(B)$ admits a unique continuous trace τ and a unique continuous determinant $\det$ (see [**Pie**, 1.7.13, 4.7.8, 4.7.11]), related for a fixed $L \in N_p(B)$ by

$$\det(I - zL) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\tau(L^n)\right), \tag{8}$$

for all $z \in \mathbb{C}$ in a suitable neighbourhood of 0 (see [**Pie**, 4.6.2]). Moreover, both τ and $\det$ are spectral, which means that $\tau(L) = \sum_{n=1}^{\infty}\lambda_n(L)$ and that, counting multiplicities, the zeros of the entire function $z \mapsto \det(I - zL)$ are precisely the reciprocals of the eigenvalues of L (see [**Pie**, 4.7.14, 4.7.15]).

DEFINITION 4.1. To any holomorphic map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$, the associated *dynamical determinant* is the entire function $\Delta : \mathbb{C} \to \mathbb{C}$, defined for all z of sufficiently small modulus by

$$\Delta(z) = \exp\left(-\sum_{n\in\mathbb{N}}\frac{z^n}{n}\sum_{\underline{i}\in\mathcal{I}^n}\frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))}\right), \tag{9}$$

where $w_{\underline{i}} := \prod_{k=1}^{n} w_{i_k}\circ T_{P_{k-1}\underline{i}}$, $P_k : \mathcal{I}^n \to \mathcal{I}^k$ denotes the projection $P_k\underline{i} = (i_1, \ldots, i_k)$ with the convention that $T_{P_0\underline{i}} = \mathrm{id}$, and $z_{\underline{i}}$ denotes the (unique, by [**EH**]) fixed-point of $T_{\underline{i}}$ in D.

Ruelle [**Rue1**] showed that Δ is the determinant of the strongly nuclear operator $\mathcal{L} : U(D) \to U(D)$. Therefore, if the zeros $z_1, z_2, \ldots$ of Δ are listed according to increasing modulus and counting multiplicity, then the reciprocal sequence $\{z_n^{-1}\}_{n=1}^{\infty}$ is precisely the Ruelle eigenvalue sequence.

THEOREM 4.2. *Let $(T_i, w_i, D)_{i\in\mathcal{I}}$ be a holomorphic map-weight system. Then the associated transfer operator preserves every favourable space of holomorphic functions on D, and its determinant on each of these spaces is precisely the dynamical determinant Δ.*

Proof. Comparison of (8) and (9) means we require the trace formula†

$$\tau(\mathcal{L}_A^n) = \sum_{\underline{i}\in\mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))} \quad \text{for all } n \geq 1\,, \tag{10}$$

for every favourable space A on the admissible domain D.

First consider the holomorphic map-weight system (T, w, D) consisting of a single map and weight. Since $T(D) \subset\subset D$, the Earle-Hamilton theorem [**EH**] implies that T has a unique fixed-point $z_0 \in D$, and the eigenvalues of $T'(z_0)$ lie in the open unit disc [**May2**, Thm. 1]. If $\mathcal{L}f = w \cdot f \circ T$ is the corresponding transfer operator, we claim that

$$\tau(\mathcal{L}_A) = \frac{w(z_0)}{\det(I - T'(z_0))}\,. \tag{11}$$

The admissibility of D and favourability of $A = A(D)$ are invariant under affine coordinate changes, and τ is invariant under continuous similarities, so we may assume that $z_0 = 0$ and $\|T'(0)\|_{L(\mathbb{C}^d)} < 1$. Therefore, by Lemma 2.4 and Remark 2.5, there exists $R > 0$ such that, for $r \in (0, R)$, the radius-r Euclidean ball B_r centred at 0 is admissible.

Let $H_r^2 = H^2(B_r)$ denote the Hardy space on B_r, a favourable Hilbert space (see Remark 2.8) with inner product $(f,g)_{H_r^2} = \int_{S_r} f^*\, \overline{g^*}\, d\sigma_r$, where $S_r = \partial B_r$, $\sigma_r(S_r) = 1$, and with orthonormal basis (cf. [**Rud**, Prop. 1.4.8, 1.4.9]) $\left\{p_{\underline{n},r} \mid \underline{n} \in \mathbb{N}_0^d\right\}$, where $p_{\underline{n},r}(z) = K_{\underline{n}} r^{-|\underline{n}|} z^{\underline{n}}$ and $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)!\,\underline{n}!}}$, $\underline{n} = (n_1, \ldots, n_d)$, $z^{\underline{n}} = z_1^{n_1} \cdots z_d^{n_d}$, $\underline{n}! = n_1! \cdots n_d!$, $|\underline{n}| = n_1 + \cdots + n_d$.

The canonical embedding $J_r : A \hookrightarrow H_r^2$ has dense range, because complex polynomials are dense in H_r^2, and $J_r \mathcal{L}_A = \mathcal{L}_{H_r^2} J_r$. An intertwining argument of Grabiner [**Gra**, Lem. 2.3] then implies that $\lambda(\mathcal{L}_A) = \lambda(\mathcal{L}_{H_r^2})$, and hence that $\tau(\mathcal{L}_A) = \tau(\mathcal{L}_{H_r^2})$ because τ is spectral. The strong nuclearity of $\mathcal{L}_{H_r^2}$ means it is trace class, so $\tau(\mathcal{L}_{H_r^2})$ equals the sum of the diagonal entries of the matrix representation of $\mathcal{L}_{H_r^2}$ with respect to an orthonormal basis. Thus, for any $r \in (0, R)$,

$$\begin{aligned}\tau(\mathcal{L}_A) = \tau(\mathcal{L}_{H_r^2}) &= \sum_{\underline{n}\in\mathbb{N}_0^d} (\mathcal{L}p_{\underline{n},r}, p_{\underline{n},r})_{H_r^2} = \int_{S_r} w(z) \sum_{\underline{n}\in\mathbb{N}_0^d} K_{\underline{n}}^2 r^{-2|\underline{n}|} T(z)^{\underline{n}}\, \overline{z}^{\underline{n}}\, d\sigma_r(z) \\ &= \int_{S_r} \frac{w(z)}{\left(1 - (r^{-1}T(z), r^{-1}z)_{\mathbb{C}^d}\right)^d}\, d\sigma_r(z) = \int_{S_1} \frac{w(rz)}{\left(1 - (r^{-1}T(rz), z)_{\mathbb{C}^d}\right)^d}\, d\sigma_1(z)\,.\end{aligned}$$

Letting $r \to 0$ gives

$$\tau(\mathcal{L}_A) = \int_{S_1} \frac{w(0)}{\left(1 - (T'(0)z, z)_{\mathbb{C}^d}\right)^d}\, d\sigma_1(z) = \frac{w(0)}{\det(I - T'(0))}$$

by an elementary integration, and (11) is proved.

Returning to the case of the holomorphic map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$, the factorisation argument used in the proof of Proposition 2.10 shows that for $n \in \mathbb{N}$,

† This trace formula (10) generalises the original one of Ruelle [**Rue1**] for $A = U(D)$, as well as that of Mayer [**May1, May2, May3**]. Our method of proof is rather direct, reducing to a simple Hilbert space computation; in particular, we do not need to explicitly evaluate the eigenvalues of each weighted composition operator $f \mapsto w_{\underline{i}} \cdot f \circ T_{\underline{i}}$ (a more complicated procedure, particularly in higher dimensions, cf. [**May2**, §III]).

the series $\sum_{\underline{i}\in\mathcal{I}^n}\mathcal{L}_{\underline{i}}$ converges in $N_{2/3}(A)$ to $\mathcal{L}_A^n$, where $\mathcal{L}_{\underline{i}}: A \to A$ is given by $\mathcal{L}_{\underline{i}}f = w_{\underline{i}}\cdot f\circ T_{\underline{i}}$. Since τ is continuous, $\tau(\mathcal{L}_A^n) = \sum_{\underline{i}\in\mathcal{I}^n}\tau(\mathcal{L}_{\underline{i}})$, and the required trace formula (10) follows from (11). □

COROLLARY 4.3. *Let $(T_i, w_i)_{i\in\mathcal{I}}$ be a C^ω map-weight system such that $(T_i)_{i\in\mathcal{I}}$ is complex contracting. Then the associated transfer operator preserves every favourable space on every admissible domain, and its eigenvalue sequence on each of these spaces is precisely the Ruelle eigenvalue sequence.*

In view of Corollary 4.3, the Ruelle eigenvalue sequence associated with a complex contracting C^ω map-weight system will henceforth be denoted simply by $\lambda(\mathcal{L}) = \{\lambda_n(\mathcal{L})\}_{n=1}^\infty$.

COROLLARY 4.4. *Under the hypotheses of Theorem 3.2, the Ruelle eigenvalue sequence $\lambda(\mathcal{L})$ can be bounded by*

$$|\lambda_n(\mathcal{L})| < \min\left(n^{1/2}\,,\ \frac{\sqrt{d}}{(1-r^2)^{d/2}}\,n^{(d-1)/(2d)}\right)\frac{W}{r^d}r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}}. \tag{12}$$

Proof. Hardy space $H^2(D)$ is favourable, so Corollary 4.3 implies that $\lambda(\mathcal{L}_{H^2(D)})$ is the Ruelle eigenvalue sequence. The bound

$$|\lambda_n(\mathcal{L})| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}}\,n^{(d-1)/(2d)}\,r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}}$$

then follows from [**BJ**, Thm. 1]. The other part of (12) is immediate from Theorem 3.2. □

REMARK 4.5. For a given $(T_i, w_i)_{i\in\mathcal{I}}$, if $r < 1$ is chosen as small as possible then the part of (12) arising from [**BJ**] is asymptotically superior as $n\to\infty$. For sufficiently small n, the part of (12) arising from Theorem 3.2 is sharper. For example, in dimension $d = 1$ this latter bound on $|\lambda_n(\mathcal{L})|$ is superior whenever $n^2 < 1/(1-r^2)$; this is always the case for $n = 1$, and may be true for many n if r is large (i.e. the map system is weakly contracting).

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