

# On the Ruelle eigenvalue sequence

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(Received ??)

*Abstract.* For certain real analytic data, we show that the eigenvalue sequence of the associated transfer operator  $\mathcal{L}$  is insensitive to the holomorphic function space on which  $\mathcal{L}$  acts. Explicit bounds on this eigenvalue sequence are established.

## 1. Introduction

For compact  $X \subset \mathbb{C}^d$ , and appropriate real analytic  $T_i : X \rightarrow X$  and  $w_i : X \rightarrow \mathbb{C}$ , Ruelle [Rue1] considered the action of the transfer operator  $\mathcal{L}f := \sum_i w_i \cdot f \circ T_i$  on  $U(D)$ , where  $D \subset \mathbb{C}^d$  is a domain on which all the  $T_i$ 's and  $w_i$ 's are holomorphic, and  $U(D)$  consists of those holomorphic functions on  $D$  which extend continuously to the closure of  $D$ . Ruelle proved that  $\mathcal{L} : U(D) \rightarrow U(D)$  is nuclear, hence in particular compact, and that its eigenvalue sequence  $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$ , henceforth referred to as the *Ruelle eigenvalue sequence*, is given by the reciprocals of the zeros of a dynamical determinant  $\Delta$  (see (9) for the definition).

In view of its various interpretations and applications (e.g. correlation decay rates [Bal, CPR], Fourier resonances [Rue2], Laplacians for hyperbolic surfaces [Pol1, PR], Feigenbaum period-doubling [AAC, CCR, JMS, Pol2]), it is desirable to establish explicit bounds on the Ruelle eigenvalue sequence. In the case where  $D$  may be chosen as a ball, and the  $T_i$  all map  $D$  within the concentric ball whose radius is  $r < 1$  times that of  $D$ , we establish (Theorem 3.2) the stretched-exponential bound

$$|\lambda_n(\mathcal{L})| < \frac{W}{r^d} n^{1/2} r^{\frac{d}{d+1}} (d!)^{1/d} n^{1/d} \quad \text{for all } n \geq 1, \quad (1)$$

where  $W := \sup_{z \in D} \sum_i |w_i(z)|$ .

We go on to investigate properties of transfer operators acting on other spaces of holomorphic functions, and prove (Theorem 4.2) that the Ruelle eigenvalue sequence is in a sense *universal*: for a wide range of domains  $D$ , and a broad class of spaces  $A(D)$  of holomorphic functions on  $D$ , the eigenvalue sequence of  $\mathcal{L} : A(D) \rightarrow A(D)$  is precisely

the Ruelle eigenvalue sequence. This universality suggests the possibility of sharpening the estimate (1), by adapting the proof of Theorem 3.2 to some other space  $A(D)$ . In particular, the choice of  $A(D)$  as the Hardy space  $H^2(D)$  is known to yield a concrete eigenvalue bound for  $\mathcal{L} : H^2(D) \rightarrow H^2(D)$  (see [BJ]). Intriguingly, this bound turns out to be *complementary* to (1): in every dimension  $d$ , and for every  $r < 1$ , (1) is superior for sufficiently small  $n$ , while the Hardy space bound is superior for sufficiently large  $n$ . If  $N(r, d)$  denotes the integer such that (1) gives the sharper bound on  $|\lambda_n(\mathcal{L})|$  precisely for  $1 \leq n \leq N(r, d)$ , then both  $r \mapsto N(r, d)$  and  $d \mapsto N(r, d)$  are increasing (cf. Corollary 4.4, Remark 4.5); in other words, (1) is more useful if the  $T_i$  are weakly contracting, or if the ambient dimension is high.

## 2. Transfer operators on favourable spaces of holomorphic functions

NOTATION 2.1. Let  $\mathbb{N}$  denote the set of strictly positive integers, and set  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . For  $d \in \mathbb{N}$ , equip  $\mathbb{C}^d$  with the Euclidean inner product  $(\cdot, \cdot)_{\mathbb{C}^d}$ , the corresponding norm  $\|\cdot\|_{\mathbb{C}^d}$ , and the induced Euclidean metric, denoted  $\delta$ . For  $X \subset \mathbb{C}^d$  we use  $\Delta_\varepsilon(X) = \{z \in \mathbb{C}^d \mid \delta(z, X) < \varepsilon\}$  for the Euclidean  $\varepsilon$ -neighbourhood of  $X$ . The set of all bounded domains (non-empty connected open subsets) in  $\mathbb{C}^d$  will be denoted by  $\mathcal{D}_d$ . For two bounded open sets  $\Delta_1, \Delta_2 \subset \mathbb{C}^d$  we write  $\Delta_1 \subset\subset \Delta_2$  to mean that  $\overline{\Delta_1} \subset \Delta_2$ .

Let  $B = (B, \|\cdot\|_B)$  be a Banach space. We often write  $\|\cdot\|$  instead of  $\|\cdot\|_B$  whenever this does not lead to confusion. For  $X \subset \mathbb{C}^d$  compact and  $D \in \mathcal{D}_d$  define

$$Hol(D, B) := \{f : D \rightarrow B \mid f \text{ holomorphic}\}$$

$$C(X, B) := \{f : X \rightarrow B \mid f \text{ continuous}\}, \|f\|_{C(X, B)} := \sup_{x \in X} \|f(x)\|_B$$

$$U(D, B) := \{f : \overline{D} \rightarrow B \mid f \in C(\overline{D}, B) \cap Hol(D, B)\}, \|f\|_{U(D, B)} := \sup_{z \in \overline{D}} \|f(z)\|_B.$$

Note that  $C(X, B)$  and  $U(D, B)$  are Banach spaces when equipped with the indicated norms, while  $Hol(D, B)$  is a Fréchet space when equipped with the topology of uniform convergence on compact subsets of  $D$ . If  $(B, \|\cdot\|) = (\mathbb{C}, |\cdot|)$  then we use  $C(X)$ ,  $Hol(D)$ , and  $U(D)$  to denote  $C(X, \mathbb{C})$ ,  $Hol(D, \mathbb{C})$ , and  $U(D, \mathbb{C})$  respectively.

We use  $L(B)$  to denote the space of bounded linear operators from a Banach space  $(B, \|\cdot\|)$  to itself, always equipped with the induced operator norm.

If  $T$  is holomorphic on some  $D \in \mathcal{D}_d$ , its derivative at  $z \in D$  is denoted by  $T'(z)$ .

DEFINITION 2.2. Let  $\mathcal{I}$  be a non-empty countable set. For  $D \in \mathcal{D}_d$ , a collection  $(T_i)_{i \in \mathcal{I}} = (T_i, D)_{i \in \mathcal{I}}$  of holomorphic maps  $T_i \in U(D, \mathbb{C}^d)$  is called a *holomorphic map system* (on  $D$ ) if  $\cup_{i \in \mathcal{I}} T_i(D) \subset\subset D$ .

Write  $T_{\underline{i}} := T_{i_n} \circ \dots \circ T_{i_1}$  for  $\underline{i} = (i_1, \dots, i_n) \in \mathcal{I}^n$ ,  $n \in \mathbb{N}$ .

For  $X \subset \mathbb{C}^d$  compact, a collection  $(T_i)_{i \in \mathcal{I}} = (T_i, X)_{i \in \mathcal{I}}$  of maps  $T_i : X \rightarrow X$  is a  *$C^\omega$  map system* (on  $X$ ) if there exists  $D \in \mathcal{D}_d$  with  $X \subset\subset D$  such that each  $T_i$  extends holomorphically to  $D$  and  $(T_i, D)_{i \in \mathcal{I}}$  is a holomorphic map system. Any such  $D$  is called *admissible* for the  $C^\omega$  map system  $(T_i, X)_{i \in \mathcal{I}}$ .

For  $n \in \mathbb{N}$ , a  $C^\omega$  map system  $(T_i, X)_{i \in \mathcal{I}}$  is called *complex  $n$ -contracting* (or simply *complex contracting*) if there exists  $D \in \mathcal{D}_d$  with  $X \subset\subset D$ , such that  $T_{\underline{i}}' \in U(D, L(\mathbb{C}^d))$

for every  $\underline{i} \in \mathcal{I}^n$  and

$$\sup_{\underline{i} \in \mathcal{I}^n} \left\| \frac{T'_{\underline{i}}}{T_{\underline{i}}} \right\|_{U(D, L(\mathbb{C}^d))} < 1. \quad (2)$$

Note that if  $\mathcal{I}$  is finite then (2) is implied by the condition  $\sup_{\underline{i} \in \mathcal{I}^n} \|T'_{\underline{i}}\|_{C(X, L(\mathbb{C}^d))} < 1$ .

EXAMPLE 2.3. If  $X = [0, 1] \subset \mathbb{C}$ , define the *Gauss map system*  $(T_i)_{i \in \mathbb{N}}$  by  $T_i(x) = 1/(i+x)$  (the  $T_i$  are the inverse branches to the Gauss map  $x \mapsto 1/x \pmod{1}$  on  $X$ ). This is a  $C^\omega$  map system on  $X$ : for example if  $D \subset \mathbb{C}$  is the open disc of radius  $3/2$  centred at the point 1 then  $(T_i, D)_{i \in \mathcal{I}}$  is a holomorphic map system. The system is also complex contracting, because  $\sup_{\underline{i} \in \mathcal{I}^2} \|T'_{\underline{i}}\|_{U(D)} = |T'_{(1,1)}(-1/2)| = 4/9 < 1$  (note we cannot choose  $n = 1$  in (2), because  $T'_1(0) = -1$ ).

Complex contraction guarantees the existence of an admissible domain, and this domain may be chosen arbitrarily close to  $X$ :

LEMMA 2.4. *If a  $C^\omega$  map system on  $X$  is complex contracting then there exists a family  $\{D_\theta\}_{\theta \in (0, \Theta)}$  of admissible domains, such that  $\bigcap_{\theta \in (0, \Theta)} D_\theta = X$ .*

*Proof.* Let  $(T_i)_{i \in \mathcal{I}}$  denote the  $C^\omega$  map system on  $X$ . Choose  $n \in \mathbb{N}$  and  $D \in \mathcal{D}_d$  such that  $\gamma := \sup_{\underline{i} \in \mathcal{I}^n} \|T'_{\underline{i}}\|_{U(D, L(\mathbb{C}^d))} < 1$ . From the several variables mean value theorem [Ave, Thm. 2.3], for each  $\underline{i} \in \mathcal{I}^n$ , the map  $T_{\underline{i}}$  is  $\gamma$ -Lipschitz, with respect to Euclidean distance  $\delta$ , on any convex subset of  $D$ . Now set  $\beta := \gamma^{1/n} < 1$ , and define the distance

$$\text{dist}(x, y) = \sup_{\underline{i} \in \mathcal{I}^{n-1}} \sum_{k=0}^{n-1} \beta^{n-1-k} \delta(T_{P_k(\underline{i})}(x), T_{P_k(\underline{i})}(y)),$$

where for  $1 \leq k \leq n-1$ ,  $P_k : \mathcal{I}^{n-1} \rightarrow \mathcal{I}^k$  denotes the projection  $P_k \underline{i} = (i_1, \dots, i_k)$  onto the first  $k$  coordinates, with the convention that  $T_{P_0 \underline{i}} = \text{id}$ . Note that for each  $i \in \mathcal{I}$ , the map  $T_i$  is  $\beta$ -Lipschitz, with respect to  $\text{dist}$ , on any convex subset of  $D$ . Moreover,  $\text{dist}$  and  $\delta$  generate the same topology on  $\delta$ -compact subsets of  $D$ . To see this observe that on the one hand we clearly have  $\delta(x, y) \leq \beta^{1-n} \text{dist}(x, y)$  for every  $x, y \in D$ . On the other hand, if  $K$  is a  $\delta$ -compact convex subset of  $D$ , then by Cauchy's theorem there is  $C > 0$  such that  $\|T'_{\underline{i}}\|_{C(K, L(\mathbb{C}^d))} \leq C$  for every  $\underline{i} \in \mathcal{I}^k$ ,  $1 \leq k \leq n-1$ . Thus by the mean value theorem  $\text{dist}(x, y) \leq C \sum_{k=0}^{n-1} \beta^{n-1-k} \delta(x, y)$  for every  $x, y \in K$ .

Since  $X$  is compactly contained in the domain  $D$ , there exists  $\varepsilon > 0$  such that the Euclidean neighbourhood  $\Delta_\varepsilon(X)$  is contained in  $D$ . Setting  $\Theta := \varepsilon \beta^{n-1}$ , we see that  $D_\theta := \{z \in \mathbb{C}^d \mid \text{dist}(z, X) < \theta\} \subset \Delta_\varepsilon(X)$  for all  $\theta \in (0, \Theta]$ , and that  $\bigcap_{\theta \in (0, \Theta]} D_\theta = X$ . If  $z \in D_\theta$ , and  $x \in X$  satisfies  $\text{dist}(z, X) = \text{dist}(z, x)$ , then  $x, z \in \Delta_\varepsilon(x)$ , a convex subset of  $D$ , so  $\text{dist}(T_i(z), T_i(x)) \leq \beta \text{dist}(z, x)$ . Therefore  $\text{dist}(T_i(z), X) \leq \text{dist}(T_i(z), T_i(x)) \leq \beta \text{dist}(z, x) = \beta \text{dist}(z, X)$ , and hence  $\bigcup_{i \in \mathcal{I}} T_i(D_\theta) \subset D_{\beta\theta} \subset D_\theta$ , so  $D_\theta$  is admissible for  $(T_i, X)_{i \in \mathcal{I}}$ .  $\square$

REMARK 2.5. The above proof shows that if a  $C^\omega$  map system on  $X$  is complex 1-contracting then all sufficiently small Euclidean  $\varepsilon$ -neighbourhoods  $\Delta_\varepsilon(X)$  are admissible. This is not the case for the Gauss map system  $T_i(z) = 1/(z+i)$  on  $X = [0, 1] \subset \mathbb{C}$ : no Euclidean  $\varepsilon$ -neighbourhood is admissible, since  $\delta(T_1(-\varepsilon), X) > \varepsilon$ .

DEFINITION 2.6. Let  $\mathcal{I}$  be a non-empty countable set. A *holomorphic weight system* on  $D \in \mathcal{D}_d$  is a collection  $(w_i)_{i \in \mathcal{I}} = (w_i, D)_{i \in \mathcal{I}}$  of holomorphic functions (called *weight functions*)  $w_i \in U(D)$  such that  $\sum_{i \in \mathcal{I}} \|w_i\|_{U(D)} < \infty$ .

For  $X \subset \mathbb{C}^d$  compact, a collection  $(w_i)_{i \in \mathcal{I}} = (w_i, X)_{i \in \mathcal{I}}$  of maps  $w_i : X \rightarrow \mathbb{C}$  is a  $C^\omega$  *weight system* (on  $X$ ) if there exists  $D \in \mathcal{D}_d$  with  $X \subset D$  such that  $(w_i, D)_{i \in \mathcal{I}}$  is a holomorphic weight system. Any such  $D$  is called *admissible* for  $(w_i, X)_{i \in \mathcal{I}}$ .

If  $(T_i)_{i \in \mathcal{I}}$  is a holomorphic (respectively,  $C^\omega$ ) map system and  $(w_i)_{i \in \mathcal{I}}$  is a holomorphic (respectively,  $C^\omega$ ) weight system then  $(T_i, w_i)_{i \in \mathcal{I}}$  is called a *holomorphic (respectively,  $C^\omega$ ) map-weight system*. A domain  $D \in \mathcal{D}_d$  is called *admissible* for a  $C^\omega$  map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$  if it is admissible for both  $(T_i, X)_{i \in \mathcal{I}}$  and  $(w_i, X)_{i \in \mathcal{I}}$ .

With each holomorphic map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$  we associate a linear operator,

$$\mathcal{L}f = \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i, \quad (3)$$

called the *transfer operator*. It will be seen that the transfer operator  $\mathcal{L}$  preserves, and acts compactly upon, the following class of spaces of holomorphic functions.

DEFINITION 2.7. For  $D \in \mathcal{D}_d$ , a Banach space  $A = A(D)$  of functions  $f : D \rightarrow \mathbb{C}$  holomorphic on  $D$  is called a *favourable space of holomorphic functions (on  $D$ )* if

- (i) for each  $z \in D$ , the point evaluation functional  $f \mapsto f(z)$  is continuous on  $A$ , and
- (ii)  $A$  contains  $U(D)$ , and the natural embedding<sup>†</sup>  $U(D) \hookrightarrow A$  has norm 1.

REMARK 2.8. Let  $D \in \mathcal{D}_d$ . Then  $U(D)$  is trivially a favourable space of holomorphic functions. Other examples include, for  $p \in [1, \infty]$ , *Bergman spaces*  $L^p_{Hol}(D)$  (see [Ran, Ch. I, Cor. 1.7, 1.10]) and *Hardy spaces*  $H^p(D)$  (see [Kra, Ch. 8.3]). If  $p = 2$  and  $D$  has  $C^2$  boundary, then  $H^2(D)$  can be identified with the  $L^2(\partial D, \sigma)$ -closure of  $U(D)$ , where  $\sigma$  denotes  $(2d - 1)$ -dimensional Lebesgue on the boundary  $\partial D$ , normalised so that  $\sigma(\partial D) = 1$ . In particular,  $H^2(D)$  is a Hilbert space with inner product given by  $(f, g) = \int_{\partial D} f^* \overline{g^*} d\sigma$ , where, for  $h \in H^2(D)$ , the symbol  $h^*$  denotes the corresponding nontangential limit function in  $L^2(\partial D, \sigma)$  — see [Kra, Ch. 1.5 and 8].

Recall (see e.g. [Pie, 1.7.1]) that a linear operator  $L : B \rightarrow B$  on a Banach space<sup>†</sup>  $B$  is *p-nuclear* if there exist sequences  $b_i \in B$  and  $l_i \in B^*$  (the strong dual of  $B$ ) with  $\sum_i (\|b_i\| \|l_i\|)^p < \infty$ , such that  $L(b) = \sum_{i=1}^{\infty} l_i(b)b_i$  for all  $b \in B$ . The operator is *strongly nuclear* (or *nuclear of order zero*) if it is *p-nuclear* for every  $p > 0$ . It turns out that certain natural embeddings between favourable spaces are strongly nuclear:

LEMMA 2.9. *Let  $D$  and  $D'$  be domains in  $\mathbb{C}^d$  such that  $D' \subset D$ . Let  $A$  and  $A'$  be favourable Banach spaces of holomorphic functions on  $D$  and  $D'$  respectively. Then  $A \subset A'$ , and the natural embedding  $J : A \hookrightarrow A'$ , defined by  $Jf = f|_{D'}$ , is strongly nuclear.*

<sup>†</sup> The embedding  $U(D) \hookrightarrow A$  is automatically continuous: continuity of point evaluation on both  $A$  and  $U(D)$  implies that it has closed graph; cf. the proof of Lemma 2.9.

<sup>†</sup> See [Gro, II, Déf. 1, p. 3] for the generalisation to locally convex spaces.

*Proof.* Choose  $D'' \in \mathcal{D}_d$  with  $D' \subset D'' \subset D$ , and consider the natural embeddings

$$A \xrightarrow{J_1} \text{Hol}(D'') \xrightarrow{J_2} U(D') \xrightarrow{J_3} A'.$$

Clearly  $J = J_3 J_2 J_1$ . The unit ball of  $U(D')$  is a neighbourhood in  $\text{Hol}(D'')$ , so the map  $J_2$  is bounded. But the Fréchet space  $\text{Hol}(D'')$  is nuclear [Gro, II, Cor., p. 56], so  $J_2$  is  $p$ -nuclear for every  $p > 0$  by [Gro, II, Cor. 4, p. 39, Cor. 2, p. 61]. It thus suffices to show that  $J_1$  and  $J_3$  are continuous by [Gro, I, p. 84, II, p. 9].

Now,  $J_3$  is continuous since  $A'$  is favourable. Finally, to see that  $J_1$  is continuous we note that, by the closed graph theorem (see e.g. [Scha, Ch. III, 2.3]), it is enough to show that if  $f_n \rightarrow f$  in  $A$ , and  $J_1 f_n \rightarrow g$  in  $\text{Hol}(D'')$ , then  $g = J_1 f = f|_{D''}$ . Since point evaluation is continuous on  $A$ ,  $f_n(z) \rightarrow f(z)$  for all  $z \in D$  and in particular for all  $z \in D''$ . But point evaluation is also continuous on  $\text{Hol}(D'')$ , so  $f_n(z) = J_1 f_n(z) \rightarrow g(z)$  as  $n \rightarrow \infty$  for all  $z \in D''$ . Therefore  $g = f|_{D''}$ .  $\square$

Favourable spaces  $A$  are always invariant under the transfer operator  $\mathcal{L}$ , and the restricted operator (henceforth denoted by  $\mathcal{L}_A$ ) is always compact, indeed strongly nuclear:

**PROPOSITION 2.10.** *Let  $(T_i, w_i, D)_{i \in \mathcal{I}}$  be a holomorphic map-weight system. The corresponding transfer operator leaves invariant every favourable space  $A$  of holomorphic functions on  $D$ , and  $\mathcal{L}_A : A \rightarrow A$  is strongly nuclear.*

*Proof.* Choose  $D' \in \mathcal{D}_d$  with  $\cup_{i \in \mathcal{I}} T_i(D) \subset D' \subset D$ . First we observe that  $\hat{\mathcal{L}}f := \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$  defines a continuous operator  $\hat{\mathcal{L}} : U(D') \rightarrow U(D)$ . To see this, fix  $f \in U(D')$  and note that  $w_i \cdot f \circ T_i \in U(D)$  with  $\|w_i \cdot f \circ T_i\|_{U(D)} \leq \|w_i\|_{U(D)} \|f\|_{U(D')}$  for every  $i \in \mathcal{I}$ . But since  $\|\hat{\mathcal{L}}f\|_{U(D)} \leq \sum_{i \in \mathcal{I}} \|w_i\|_{U(D)} \|f\|_{U(D')}$  and  $\sum_{i \in \mathcal{I}} \|w_i\|_{U(D)} < \infty$  by hypothesis, we conclude that  $\hat{\mathcal{L}}f \in U(D)$  and that  $\hat{\mathcal{L}}$  is continuous.

Since  $A$  is favourable, the embedding  $\hat{J} : U(D) \hookrightarrow A$  is continuous, and the embedding  $J : A \hookrightarrow U(D')$  is  $p$ -nuclear for every  $p > 0$  by Lemma 2.9. Moreover, if  $f \in A$  then  $\mathcal{L}f = \hat{J} \hat{\mathcal{L}} J f \in A$ . Thus  $A$  is  $\mathcal{L}$ -invariant, and the operator  $\mathcal{L}_A = \hat{J} \hat{\mathcal{L}} J$  is  $p$ -nuclear for any  $p > 0$ .  $\square$

**REMARK 2.11.** Strong nuclearity of the transfer operator on spaces of holomorphic functions is not new (the original result of this kind is [Rue1], but see also e.g. [GLZ, JP, May3]); the novelty of Proposition 2.10 is in the breadth of spaces covered.  $\ddagger$

### 3. Eigenvalue bounds

For favourable  $A$ , the compactness of  $\mathcal{L}_A$  means its spectrum consists of a countable set of eigenvalues, each with finite algebraic multiplicity, together with a possible accumulation point at 0. We wish to obtain bounds on the *eigenvalue sequence*  $\lambda(\mathcal{L}_A) := \{\lambda_n(\mathcal{L}_A)\}_{n=1}^\infty$ , i.e. the sequence of all eigenvalues of  $\mathcal{L}_A$  counting algebraic multiplicities and ordered by decreasing modulus.  $\dagger$

$\ddagger$  Actually the result can be further extended to certain locally convex spaces of holomorphic functions, including  $\text{Hol}(D)$ .

$\dagger$  By convention distinct eigenvalues with the same modulus can be written in any order (see e.g. [Pie, 3.2.20]).

If  $L : B_1 \rightarrow B_2$  is a continuous operator between Banach spaces then for  $k \geq 1$ , its  $k$ -th approximation number  $a_k(L)$  is defined as

$$a_k(L) = \inf \{ \|L - K\| \mid K : B_1 \rightarrow B_2 \text{ linear with } \text{rank}(K) < k \} .$$

PROPOSITION 3.1. For a  $C^\omega$  map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$  such that  $(T_i)_{i \in \mathcal{I}}$  is complex contracting, and a favourable space  $A = A(D)$  such that  $D \in \mathcal{D}_d$  is admissible,

$$|\lambda_n(\mathcal{L}_A)| \leq W n^{1/2} \prod_{k=1}^n a_k(J)^{1/n} \quad \text{for all } n \geq 1, \quad (4)$$

where  $W := \sup_{z \in D} \sum_{i \in \mathcal{I}} |w_i(z)|$ ,  $D' \in \mathcal{D}_d$  is such that  $\cup_{i \in \mathcal{I}} T_i(D) \subset D' \subset D$ , and  $J : A(D) \hookrightarrow U(D')$  is the canonical embedding.

*Proof.* Since  $A(D)$  is favourable, the embedding  $\hat{J} : U(D) \hookrightarrow A(D)$  is continuous of norm 1. Next observe that  $\hat{\mathcal{L}}f = \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$  defines a continuous operator  $\hat{\mathcal{L}} : U(D') \rightarrow U(D)$  (see the proof of Proposition 2.10) with  $\|\hat{\mathcal{L}}\| \leq W$ . To see the latter note that for  $f \in U(D')$  we have  $|f(T_i(z))| \leq \|f\|_{U(D')}$  for every  $z \in D$ ,  $i \in \mathcal{I}$ ; thus by the maximum principle  $\|\hat{\mathcal{L}}f\|_{U(D)} = \sup_{z \in D} |(\hat{\mathcal{L}}f)(z)| \leq \sup_{z \in D} \sum_{i \in \mathcal{I}} |w_i(z)| |f(T_i(z))| \leq W \|f\|_{U(D')}$ .

Now clearly  $\mathcal{L}_A = \hat{J}\hat{\mathcal{L}}J$ , so

$$a_k(\mathcal{L}_A) \leq \|\hat{J}\hat{\mathcal{L}}\| a_k(J) \leq W a_k(J) \quad \text{for all } k \geq 1, \quad (5)$$

since in general  $a_k(L_1 L_2) \leq \|L_1\| a_k(L_2)$  whenever  $L_1$  and  $L_2$  are bounded operators between Banach spaces (see [Pie, 2.2]). Moreover, since  $\mathcal{L}_A$  is compact, Weyl's inequality (see e.g. [Hin]) asserts that  $\prod_{k=1}^n |\lambda_k(\mathcal{L}_A)| \leq n^{n/2} \prod_{k=1}^n a_k(\mathcal{L}_A)$  for every  $n \in \mathbb{N}$ .<sup>‡</sup>

Together with (5) this yields (4), because  $|\lambda_n(\mathcal{L}_A)| \leq \prod_{k=1}^n |\lambda_k(\mathcal{L}_A)|^{1/n}$ .  $\square$

Taking  $A(D) = U(D)$ , the Ruelle eigenvalue sequence  $\lambda(\mathcal{L}_{U(D)})$  can be bounded as follows:

THEOREM 3.2. Suppose the Euclidean ball  $D \subset \mathbb{C}^d$  is an admissible domain for a  $C^\omega$  map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$ , and that  $\cup_{i \in \mathcal{I}} T_i(D)$  is contained in the concentric ball whose radius is  $r < 1$  times that of  $D$ . Setting  $W := \sup_{z \in B} \sum_{i \in \mathcal{I}} |w_i(z)|$ , the Ruelle eigenvalue sequence  $\lambda(\mathcal{L}_{U(D)})$  can be bounded by

$$|\lambda_n(\mathcal{L}_{U(D)})| < \frac{W}{r^d} n^{1/2} r^{\frac{d}{d+1}} (d!)^{1/d} n^{1/d} \quad \text{for all } n \geq 1. \quad (6)$$

If  $d = 1$  then

$$|\lambda_n(\mathcal{L}_{U(D)})| \leq W n^{1/2} r^{(n-1)/2} \quad \text{for all } n \geq 1. \quad (7)$$

*Proof.* Without loss of generality let  $D = D_1$  be the open unit ball, and let the smaller concentric ball be  $D_r$ , the ball of radius  $r$  centred at 0. Let  $J : U(D_1) \hookrightarrow U(D_r)$  be the canonical embedding. From [Far, Prop. 2.1 (a)] it follows that  $a_l(J) \leq r^{t_l}$ , where  $t_l := k$  for  $\binom{k-1+d}{d} < l \leq \binom{k+d}{d}$ , hence  $\prod_{l=1}^n a_l(J)^{1/n} \leq r^{\frac{1}{n} \sum_{i=1}^n t_i}$ . If  $d = 1$  then

<sup>‡</sup> This is a Banach space version of Weyl's original inequality in Hilbert space (see [Wey]). Note that the constant  $n^{n/2}$  is optimal (see [Hin]).

$\frac{1}{n} \sum_{l=1}^n t_l = \frac{1}{n} \sum_{l=1}^n (l-1) = (n-1)/2$ , and (7) follows from (4). More generally  $t_l \geq (d!)^{1/d} l^{1/d} - d$ , so that

$$\frac{1}{n} \sum_{l=1}^n t_l \geq -d + (d!)^{1/d} \frac{1}{n} \sum_{l=1}^n l^{1/d} > -d + (d!)^{1/d} \frac{d}{d+1} n^{1/d}$$

using the estimate  $\sum_{l=1}^n l^{1/d} > \int_{x=0}^n x^{1/d} = \frac{d}{d+1} n^{1+1/d}$ , and (6) follows from (4).  $\square$

#### 4. Universality of the Ruelle eigenvalue sequence

If  $(T_i, w_i)_{i \in \mathcal{I}}$  is a  $C^\omega$  map-weight system with complex contracting  $(T_i)_{i \in \mathcal{I}}$  then, in view of Lemma 2.4 and Proposition 2.10, there is some freedom in the choice of an admissible  $D$ , and a favourable space  $A = A(D)$  on which to consider the transfer operator  $\mathcal{L}_A$ . The purpose of this section is to show that the eigenvalue sequence of  $\mathcal{L}_A$  is in fact independent of  $A$ : it is always equal to the Ruelle eigenvalue sequence  $\lambda(\mathcal{L}_{U(D)})$  (see Corollary 4.3). For this, we first require some facts from the Fredholm theory originally developed by Grothendieck [Gro]. If  $B$  is a Banach space, we denote by  $N_p(B)$  ( $p > 0$ ) the quasi-Banach operator ideal of  $p$ -nuclear operators on  $B$  (cf. [Pie, D.1.4, 1.7.1]). If  $p \leq 2/3$  then  $N_p(B)$  admits a unique continuous trace  $\tau$  and a unique continuous determinant  $\det$  (see [Pie, 1.7.13, 4.7.8, 4.7.11]), related for a fixed  $L \in N_p(B)$  by

$$\det(I - zL) = \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \tau(L^n) \right), \quad (8)$$

for all  $z \in \mathbb{C}$  in a suitable neighbourhood of 0 (see [Pie, 4.6.2]). Moreover, both  $\tau$  and  $\det$  are spectral, which means that  $\tau(L) = \sum_{n=1}^{\infty} \lambda_n(L)$  and that, counting multiplicities, the zeros of the entire function  $z \mapsto \det(I - zL)$  are precisely the reciprocals of the eigenvalues of  $L$  (see [Pie, 4.7.14, 4.7.15]).

DEFINITION 4.1. To any holomorphic map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$ , the associated *dynamical determinant* is the entire function  $\Delta : \mathbb{C} \rightarrow \mathbb{C}$ , defined for all  $z$  of sufficiently small modulus by

$$\Delta(z) = \exp \left( - \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{\underline{i} \in \mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))} \right), \quad (9)$$

where  $w_{\underline{i}} := \prod_{k=1}^n w_{i_k} \circ T_{P_{k-1}\underline{i}}$ ,  $P_k : \mathcal{I}^n \rightarrow \mathcal{I}^k$  denotes the projection  $P_k \underline{i} = (i_1, \dots, i_k)$  with the convention that  $T_{P_0 \underline{i}} = \text{id}$ , and  $z_{\underline{i}}$  denotes the (unique, by [EH]) fixed-point of  $T_{\underline{i}}$  in  $D$ .

Ruelle [Rue1] showed that  $\Delta$  is the determinant of the strongly nuclear operator  $\mathcal{L} : U(D) \rightarrow U(D)$ . Therefore, if the zeros  $z_1, z_2, \dots$  of  $\Delta$  are listed according to increasing modulus and counting multiplicity, then the reciprocal sequence  $\{z_n^{-1}\}_{n=1}^{\infty}$  is precisely the Ruelle eigenvalue sequence.

THEOREM 4.2. *Let  $(T_i, w_i, D)_{i \in \mathcal{I}}$  be a holomorphic map-weight system. Then the associated transfer operator preserves every favourable space of holomorphic functions on  $D$ , and its determinant on each of these spaces is precisely the dynamical determinant  $\Delta$ .*

*Proof.* Comparison of (8) and (9) means we require the trace formula<sup>†</sup>

$$\tau(\mathcal{L}_A^n) = \sum_{\underline{i} \in \mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))} \quad \text{for all } n \geq 1, \quad (10)$$

for every favourable space  $A$  on the admissible domain  $D$ .

First consider the holomorphic map-weight system  $(T, w, D)$  consisting of a single map and weight. Since  $T(D) \subset\subset D$ , the Earle-Hamilton theorem [EH] implies that  $T$  has a unique fixed-point  $z_0 \in D$ , and the eigenvalues of  $T'(z_0)$  lie in the open unit disc [May2, Thm. 1]. If  $\mathcal{L}f = w \cdot f \circ T$  is the corresponding transfer operator, we claim that

$$\tau(\mathcal{L}_A) = \frac{w(z_0)}{\det(I - T'(z_0))}. \quad (11)$$

The admissibility of  $D$  and favourability of  $A = A(D)$  are invariant under affine coordinate changes, and  $\tau$  is invariant under continuous similarities, so we may assume that  $z_0 = 0$  and  $\|T'(0)\|_{L(\mathbb{C}^d)} < 1$ . Therefore, by Lemma 2.4 and Remark 2.5, there exists  $R > 0$  such that, for  $r \in (0, R)$ , the radius- $r$  Euclidean ball  $B_r$  centred at 0 is admissible.

Let  $H_r^2 = H^2(B_r)$  denote the Hardy space on  $B_r$ , a favourable Hilbert space (see Remark 2.8) with inner product  $(f, g)_{H_r^2} = \int_{S_r} f^* \bar{g} \, d\sigma_r$ , where  $S_r = \partial B_r$ ,  $\sigma_r(S_r) = 1$ , and with orthonormal basis (cf. [Rud, Prop. 1.4.8, 1.4.9])  $\{p_{\underline{n}, r} \mid \underline{n} \in \mathbb{N}_0^d\}$ , where  $p_{\underline{n}, r}(z) = K_{\underline{n}} r^{-|\underline{n}|} z^{\underline{n}}$  and  $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)! \underline{n}!}}$ ,  $\underline{n} = (n_1, \dots, n_d)$ ,  $z^{\underline{n}} = z_1^{n_1} \cdots z_d^{n_d}$ ,  $\underline{n}! = n_1! \cdots n_d!$ ,  $|\underline{n}| = n_1 + \cdots + n_d$ .

The canonical embedding  $J_r : A \hookrightarrow H_r^2$  has dense range, because complex polynomials are dense in  $H_r^2$ , and  $J_r \mathcal{L}_A = \mathcal{L}_{H_r^2} J_r$ . An intertwining argument of Grabiner [Gra, Lem. 2.3] then implies that  $\lambda(\mathcal{L}_A) = \lambda(\mathcal{L}_{H_r^2})$ , and hence that  $\tau(\mathcal{L}_A) = \tau(\mathcal{L}_{H_r^2})$  because  $\tau$  is spectral. The strong nuclearity of  $\mathcal{L}_{H_r^2}$  means it is trace class, so  $\tau(\mathcal{L}_{H_r^2})$  equals the sum of the diagonal entries of the matrix representation of  $\mathcal{L}_{H_r^2}$  with respect to an orthonormal basis. Thus, for any  $r \in (0, R)$ ,

$$\begin{aligned} \tau(\mathcal{L}_A) &= \tau(\mathcal{L}_{H_r^2}) = \sum_{\underline{n} \in \mathbb{N}_0^d} (\mathcal{L} p_{\underline{n}, r}, p_{\underline{n}, r})_{H_r^2} = \int_{S_r} w(z) \sum_{\underline{n} \in \mathbb{N}_0^d} K_{\underline{n}}^2 r^{-2|\underline{n}|} T(z)^{\underline{n}} \bar{z}^{\underline{n}} \, d\sigma_r(z) \\ &= \int_{S_r} \frac{w(z)}{(1 - (r^{-1}T(z), r^{-1}z)_{\mathbb{C}^d})^d} \, d\sigma_r(z) = \int_{S_1} \frac{w(rz)}{(1 - (r^{-1}T(rz), z)_{\mathbb{C}^d})^d} \, d\sigma_1(z). \end{aligned}$$

Letting  $r \rightarrow 0$  gives

$$\tau(\mathcal{L}_A) = \int_{S_1} \frac{w(0)}{(1 - (T'(0)z, z)_{\mathbb{C}^d})^d} \, d\sigma_1(z) = \frac{w(0)}{\det(I - T'(0))}$$

by an elementary integration, and (11) is proved.

Returning to the case of the holomorphic map-weight system  $(T_i, w_i)_{i \in \mathcal{I}}$ , the factorisation argument used in the proof of Proposition 2.10 shows that for  $n \in \mathbb{N}$ ,

<sup>†</sup> This trace formula (10) generalises the original one of Ruelle [Rue1] for  $A = U(D)$ , as well as that of Mayer [May1, May2, May3]. Our method of proof is rather direct, reducing to a simple Hilbert space computation; in particular, we do not need to explicitly evaluate the eigenvalues of each weighted composition operator  $f \mapsto w_{\underline{i}} \cdot f \circ T_{\underline{i}}$  (a more complicated procedure, particularly in higher dimensions, cf. [May2, §III]).

the series  $\sum_{i \in \mathcal{I}^n} \mathcal{L}_i$  converges in  $N_{2/3}(A)$  to  $\mathcal{L}_A^n$ , where  $\mathcal{L}_i : A \rightarrow A$  is given by  $\mathcal{L}_i f = w_i \cdot f \circ T_i$ . Since  $\tau$  is continuous,  $\tau(\mathcal{L}_A^n) = \sum_{i \in \mathcal{I}^n} \tau(\mathcal{L}_i)$ , and the required trace formula (10) follows from (11).  $\square$

**COROLLARY 4.3.** *Let  $(T_i, w_i)_{i \in \mathcal{I}}$  be a  $C^\omega$  map-weight system such that  $(T_i)_{i \in \mathcal{I}}$  is complex contracting. Then the associated transfer operator preserves every favourable space on every admissible domain, and its eigenvalue sequence on each of these spaces is precisely the Ruelle eigenvalue sequence.*

In view of Corollary 4.3, the Ruelle eigenvalue sequence associated with a complex contracting  $C^\omega$  map-weight system will henceforth be denoted simply by  $\lambda(\mathcal{L}) = \{\lambda_n(\mathcal{L})\}_{n=1}^\infty$ .

**COROLLARY 4.4.** *Under the hypotheses of Theorem 3.2, the Ruelle eigenvalue sequence  $\lambda(\mathcal{L})$  can be bounded by*

$$|\lambda_n(\mathcal{L})| < \min \left( n^{1/2}, \frac{\sqrt{d}}{(1-r^2)^{d/2}} n^{(d-1)/(2d)} \right) \frac{W}{r^d} r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}}. \quad (12)$$

*Proof.* Hardy space  $H^2(D)$  is favourable, so Corollary 4.3 implies that  $\lambda(\mathcal{L}_{H^2(D)})$  is the Ruelle eigenvalue sequence. The bound

$$|\lambda_n(\mathcal{L})| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}} n^{(d-1)/(2d)} r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}}$$

then follows from [BJ, Thm. 1]. The other part of (12) is immediate from Theorem 3.2.  $\square$

**REMARK 4.5.** For a given  $(T_i, w_i)_{i \in \mathcal{I}}$ , if  $r < 1$  is chosen as small as possible then the part of (12) arising from [BJ] is asymptotically superior as  $n \rightarrow \infty$ . For sufficiently small  $n$ , the part of (12) arising from Theorem 3.2 is sharper. For example, in dimension  $d = 1$  this latter bound on  $|\lambda_n(\mathcal{L})|$  is superior whenever  $n^2 < 1/(1-r^2)$ ; this is always the case for  $n = 1$ , and may be true for many  $n$  if  $r$  is large (i.e. the map system is weakly contracting).

*Acknowledgements.* We are grateful to B. Carl, H. König, V. Mascioni and A. Pietsch for advice on the sharpness of Weyl's inequality in Banach spaces, and K. Osipenko, A. Pinkus and M. Stessin for bibliographical assistance concerning the approximation numbers for the embedding of  $U(B_1)$  in  $U(B_r)$ . Both authors were partially supported by EPSRC grants GR/R64650/01 (2001–2003) and GR/S50991/01 (2004–2007), and the second author was partially supported by an EPSRC Advanced Research Fellowship.

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