Smooth Cocycle Rigidity For Expanding Maps, and an Application to Mostow Rigidity

Oliver Jenkinson

School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road,
London, E1 4NS.
email: omj@maths.qmw.ac.uk

Abstract. *We give a variation on the proof of Mostow's rigidity theorem, for certain hyperbolic 3-manifolds. This is based on a rigidity theorem for conjugacies between piecewise-conformal expanding Markov maps. The conjugacy rigidity theorem is deduced from a Livsic cocycle rigidity theorem that we prove for smooth, compact Lie group-valued cocycles over piecewise smooth expanding Markov maps.*

1991 Mathematics Subject Classification: 58F15, 58F11, 30F40, 30F45

Section 1. Introduction.

The hallmark of any kind of rigidity result is that an a priori mild condition of some sort forces an unexpectedly strong one. Perhaps the most striking result of this kind is Mostow's rigidity theorem [12], which states that, for compact connected orientable hyperbolic manifolds of dimension at least three, a mere *homotopy* equivalence in fact implies the much stronger *geometric* equivalence of isometry.

One of the aims of this paper is to show that, in certain cases, Mostow's theorem can be proved using techniques from the branch of ergodic theory known as *Livsic theory*. The central theme of Livsic theory is the consideration of solutions $w : X \to G$ to *cocycle equations* of the form

$$w(Tx)u(x) = v(x)w(x), \tag{1}$$

where $T : X \to X$ is some dynamical system, G is a Lie group, and $u, v : X \to G$ have some specified degree of regularity (Hölder, smooth, analytic, etc). The guiding philosophy is that, under suitable conditions on T (typically some kind of hyperbolicity), any measurable almost everywhere solution w of (1) is essentially as regular as u and v. This is another kind of rigidity result, a *cocycle rigidity* result.

The original cocycle rigidity results are due to Livsic [6], [7], who considered real-valued cocycles over both Anosov systems and subshifts of finite type, and proved theorems of the following kind:

Livsic's Cocycle Rigidity Theorem (Livsic, [6], [7]) *Let $T : X \to X$ be a transitive Anosov diffeomorphism on a compact Riemannian manifold X. Let m be a T-invariant probability measure, absolutely continuous with respect to Riemannian volume. Suppose $\theta : X \to \mathbb{R}$ is Hölder continuous, and the measurable function $w : X \to \mathbb{R}$ satisfies $w(Tx) - w(x) = \theta(x)$ almost everywhere. Then there exists a Hölder function $w' : X \to \mathbb{R}$ such that $w = w'$ almost everywhere, and such that $w'(Tx) - w'(x) = \theta(x)$ holds everywhere.*

Parry & Pollicott [19] considered the technically more difficult problem of proving regularity theorems for cocycles with values in certain *non-abelian* Lie groups. They proved results of the following kind:

Non-abelian Cocycle Rigidity Theorem (Parry & Pollicott, [19]) *Let $T : X \to X$ be a transitive Anosov diffeomorphism on a compact Riemannian manifold, and G a compact Lie group. Suppose the functions $u, v : X \to G$ are Hölder continuous, and the measurable function $w : X \to G$ satisfies $w(Tx)u(x) = v(x)w(x)$ almost everywhere with respect to some Hölder equilibrium state. Then there exists a Hölder function $w' : X \to G$ such that $w = w'$ almost everywhere, and such that $w'(Tx)u(x) = v(x)w'(x)$ holds everywhere.*

The essential difficulty in the non-abelian setting is that it is not immediately clear how to rewrite (1) as an equation $w(Tx) = \theta(x)w(x)$ for some function θ depending on u and v (by contrast, setting $\theta = v - u$ in Livsic's real-valued theorem does achieve this). One of the key observations of Parry & Pollicott [19] is that in fact it *is* possible to express (1) as $w(Tx) = \theta(x)w(x)$, provided all terms are suitably (re-)interpreted. This involves considering G as a subset of some vector space $\mathbb{C}^{d^2}$, and interpreting each $\theta(x)$ as a linear map acting on $\mathbb{C}^{d^2}$. Parry & Pollicott then related the equation $w(Tx) = \theta(x)w(x)$ to a fixed point equation for a certain *transfer operator* which they defined in terms of θ (and hence in terms of u, v). In particular, the solution w can be identified with an eigenfunction for the dominant eigenvalue 1, and its regularity analysed using spectral properties of the operator.

The Parry-Pollicott results are obtained firstly in the context of Hölder functions on subshifts of finite type. The above theorem for Anosov (or, more generally, Axiom A) diffeomorphisms is then deduced by using Markov partitions. As a further generalisation over the original Livsic theorem, and as a natural by-product of their method, the reference measure in the above theorem can be any Hölder equilibrium state.

Employing different techniques, Nitica & Torok [14], [15], considered regularity theorems for *smooth* cocycles over Anosov systems. These cocycles were assumed to take values in matrix Lie groups or diffeomorphism groups. A typical result is the following:

Non-abelian Smooth Cocycle Rigidity Theorem (Nitica & Torok, [15]) *Let $T : X \to X$ be a $C^{\mathbf{k}}$ Anosov diffeomorphism on a compact Riemannian manifold X, and G a matrix Lie group. Suppose $u, v : X \to G$ are $C^{\mathbf{k}}$. If the Hölder function $w : X \to G$ satisfies the cocycle equation $w(Tx)u(x) = v(x)w(x)$, then in fact w is $C^{\mathbf{k}-\varepsilon}$ for all $\varepsilon > 0$.*

More recent contributions to Livsic theory have focused on extending rigidity results to cocycles with values in non-compact groups (see in particular Nicol & Pollicott [13], Pollicott & Walkden [23], and Schmidt [28]). Many cocycle rigidity results have analogues in the context of hyperbolic flows (see in particular Walkden [30]).

Throughout this article, our dynamical system $T : X \to X$ will be a piecewise-$C^{\mathbf{k}}$ expanding Markov map, where the space X is some suitable disjoint union of regions of $\mathbb{R}^p$ (we will be more precise in §2). We fix a reference measure m (which in general can be any equilibrium state for a smooth potential, and in particular could be chosen to be absolutely continuous with respect to Lebesgue measure). Letting $U(d)$ denote the set of $d \times d$ unitary matrices, and $M(d)$ the space of $d \times d$ matrices with complex entries, we prove the following theorem (see §4 for a more precise statement):

Theorem A. *Let $T : X \to X$ be a piecewise $C^{\mathbf{k}}$ expanding Markov map. Suppose the functions $u, v : X \to U(d)$ are piecewise $C^{\mathbf{k}}$. If $w : X \to M(d)$ is an m-integrable solution* *to the cocycle equation*

$$w(Tx)u(x) = v(x)w(x) \quad a.e.\ (m),$$

then there exists a piecewise $C^{\mathbf{k}}$ *function* $w' : X \to M(d)$ *such that* $w = w'$ *almost everywhere, and* $w'(Tx)u(x) = v(x)w'(x)$ *everywhere.*

Since every compact Lie group G has a unitary representation, we deduce (see Corollary A.1) an analogous cocycle rigidity result for $C^{\mathbf{k}}$ functions u, v with values in G.

In §5 we consider the rigidity of topological conjugacies h between piecewise-conformal expanding Markov maps T_1, T_2. As for cocycle rigidity theorems, the idea is that a modest amount of regularity in fact implies some higher regularity. A topological conjugacy is *absolutely continuous* if the set $\{x : h \text{ is differentiable at } x \text{ and } h^{-1} \text{ is differentiable at } h(x)\}$ has full Lebesgue measure, and all partial derivatives of h and h^{-1} are L^2. We then prove (see §5 for a more precise statement):

Theorem B. *Any absolutely continuous conjugacy between piecewise* $C^{\mathbf{k}}$ *conformal expanding Markov maps* T_1, T_2 *is itself piecewise* $C^{\mathbf{k}}$*.*

To prove Theorem B we follow the approach of de la Llave, Marco & Moriyón [9] in relating the problem to the Livsic cocycle equation. We differentiate the conjugacy equation $hT_1 = T_2 h$ almost everywhere. This leads to a matrix-valued cocycle equation, for which the derivative Dh is an almost everywhere solution. The regularity of this solution is then analysed by means of Theorem A, combined with Livsic's original theorem.

Finally, in §6 we show how Theorem B can be used to simplify a part of Mostow's theorem. Our proof applies to those hyperbolic 3-manifolds whose fundamental groups (acting on the boundary of hyperbolic 3-space) can be modelled by Bowen-Series expanding Markov maps.

Section 2. Preliminaries.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a finite collection of subsets of Euclidean space $\mathbb{R}^p$, where each P_i is the closure of an open subset which is $C^{\mathbf{r}}$ diffeomorphic to the open unit ball. Our space $X = \sqcup_{i=1}^n P_i$ will be the corresponding disjoint union. We let ∂P_i denote the topological boundary of the piece P_i, and write $\partial\mathcal{P} = \partial P_1 \sqcup \ldots \sqcup \partial P_n$ for the disjoint union of these boundaries. We assume $m(\partial\mathcal{P}) = 0$, where our reference measure m is an equilibrium state defined below (note that in our application to Mostow's theorem, the boundaries of partition pieces will be finite unions of circular arcs, and m will be equivalent to Lebesgue measure).

Let $T : X \to X$ be a *piecewise $C^{\mathbf{r}}$ expanding Markov map.* That is,

1. T is topologically mixing,
2. for each i there exists a union $\sqcup_j P_j$ of pieces in $\mathcal{P}$ such that $T(P_i) = \sqcup_j P_j$,
3. for each i, $T|_{P_i}$ is $C^{\mathbf{r}}$,
4. T is expanding. That is, there exists $\gamma < 1$ such that for all $1 \leq i \leq n$ and all $x \in P_i$ we have $|D_xT(v)|_p \geq \gamma^{-1}|v|_p$ for all tangent vectors v, where $|\cdot|_p$ is the Euclidean norm on $\mathbb{R}^p$.

Let $C^{\mathbf{k}}(X, Y)$ denote the space of piecewise $C^{\mathbf{k}}$ functions $w : X \to Y$, for Y a subset of a finite dimensional normed vector space. By a piecewise $C^{\mathbf{k}}$ function we mean one that is $C^{\mathbf{k}}$ on each piece P_i of the partition $\mathcal{P}$, and whose appropriate directional derivatives exist on each boundary ∂P_i.

Here the regularity index $\mathbf{k} = (k, \epsilon)$, with integer $k \geq 0$ and $0 < \epsilon \leq 1$ (we say that a function is $C^{(k,\epsilon)}$ if it is C^k and if the k^{th} derivative is Hölder continuous of exponent ϵ), or $\mathbf{k} = \infty$. We write $|\mathbf{k}| = k + \epsilon$ or $|\mathbf{k}| = \infty$ accordingly.

We define the obvious lexicographic order on the regularity indices so that

$$\mathbf{k} < \mathbf{k}' \quad \Longleftrightarrow \quad C^{\mathbf{k}'}(X, Y) \subsetneq C^{\mathbf{k}}(X, Y).$$

The derivative of a piecewise $C^{\mathbf{k}}$ function, $\mathbf{k} \geq \mathbf{1}$, is a piecewise $C^{\mathbf{k}-1}$ function. Here

$$\mathbf{k} - 1 = \begin{cases} (k-1, \epsilon) & \text{if } \mathbf{k} = (k, \epsilon) \text{ is finite} \\ \infty & \text{if } \mathbf{k} = \infty. \end{cases}$$

Let $\|\cdot\|_\infty$ be the uniform norm on the space $C^{\mathbf{k}}(X, Y)$. Let $\|\cdot\|_\epsilon$ denote the ϵ-Hölder norm on the space $C^{(0,\epsilon)}(X, Y)$. That is,

$$\|w\|_\epsilon = \sup\left\{ \frac{|w(x) - w(y)|_Y}{\rho(x, y)^\epsilon} : x, y \in P_i \text{ for some } 1 \leq i \leq n,\ x \neq y \right\}.$$

If $\mathbf{k} < \infty$ then let $\| \cdot \|_k$ denote the C^k norm on the space $C^{\mathbf{k}}(X,Y)$. That is,

$$\| w \|_k = \sum_{j=0}^{k} \| D^j w \|_\infty ,$$

where $D^j w$ is the j^{th} derivative of w.

If $\mathbf{k} = (k, \epsilon) < \infty$ then let $\| \cdot \|_{\mathbf{k}}$ denote the $C^{\mathbf{k}}$ norm on the space $C^{\mathbf{k}}(X,Y)$. That is,

$$\|w\|_{\mathbf{k}} = \|w\|_k + \|D^k w\|_\epsilon .$$

For $\mathbf{k} < \infty$, $C^{\mathbf{k}}(X,Y)$ is a Banach space with respect to the $C^{\mathbf{k}}$ norm $\| \cdot \|_{\mathbf{k}}$. Henceforth, whenever we write $C^{\mathbf{k}}(X,Y)$ it will be understood that $\mathbf{k} < \infty$, and that the space is equipped with its Banach norm.

For any bounded linear operator $M : C^{\mathbf{k}}(X,Y) \to C^{\mathbf{k}}(X,Y)$ we define its operator norm by $\| M \|_{\mathbf{k}} = \sup\{\| Mw \|_{\mathbf{k}} : w \in C^{\mathbf{k}}(X,Y) \text{ with } \| w \|_{\mathbf{k}} = 1\}$.

Let m be an equilibrium state for a normalised $C^{\mathbf{s}}$ potential function $f : X \to \mathbb{R}$, where $\mathbf{s} \leq \mathbf{r}$. This means that $m = L^* m$ is the unique fixed point of the dual L^* of the (Ruelle) transfer operator $L : C^0(X, \mathbb{R}) \to C^0(X, \mathbb{R})$ defined by

$$Lw(x) = \sum_{Ty=x} e^{f(y)} w(y),$$

where L has spectral radius 1, and satisfies $L1 = 1$ (see Parry & Pollicott [18] and Ruelle [26] for more on such operators, equilibrium states, and thermodynamic formalism). In particular, the measure m is a fully supported T-invariant probability measure. In our applications to conjugacy rigidity and Mostow rigidity from section 5 onwards, we will specialise to the case where m is the (unique) T-invariant probability measure which is absolutely continuous with respect to Lebesgue measure. In this case the normalised potential function is simply $f(x) = -\log|\det(D_x T)|$.

With respect to m we define corresponding L^p-spaces $L^p(X,Y)$ with norm $\| \cdot \|_{L^p}$. Recall in particular that $\| \cdot \|_{L^1}$ also defines a norm on $C^{\mathbf{k}}(X,Y)$ for any $\mathbf{k} \geq 0$, and that $C^{\mathbf{k}}(X,Y)$ is L^1-dense in $L^1(X,Y)$. Let $\| M \|_{L^p}$ denote the norm of any bounded linear operator $M : L^p(X,Y) \to L^p(X,Y)$.

In the sequel we will consider various cocycle equations for which an L^1 function w is an almost everywhere solution. We say a $C^{\mathbf{k}}$ function w' is a $C^{\mathbf{k}}$ *version* of w if $w = w'$ almost everywhere with respect to the reference measure m. In particular, since m is fully supported (see [31]), w' will then be an *everywhere* solution of the cocycle equation.

Section 3. A Transfer Operator.

Fix the integer $d \geq 1$, and let $|\cdot|_d$ denote the Euclidean norm on $\mathbb{C}^d$, and $|\cdot|$ the absolute value on $\mathbb{C}$. Let $M(d)$ denote the space of $d \times d$ matrices with entries in $\mathbb{C}$. As a vector space, $M(d)$ is isomorphic to $\mathbb{C}^{d^2}$. Let $|\cdot|_{d^2}$ be the operator norm on $M(d)$ (and thus on $\mathbb{C}^{d^2}$) given by

$$|A|_{d^2} = \sup\{|Av|_d : v \in \mathbb{C}^d \text{ with } |v|_d = 1\}.$$

Similarly, by identifying $\mathbb{C}^{d^4}$ with the space $M(d^2)$, we define $|\cdot|_{d^4}$ to be the operator norm on $M(d^2)$. Let $< \cdot, \cdot >_d$ and $< \cdot, \cdot >_{d^2}$ denote the respective Euclidean inner products on $\mathbb{C}^d$ and $\mathbb{C}^{d^2}$ (and hence on $M(d)$). Let $U(d) = \{A \in M(d) : A^\star A = AA^\star = I\}$ denote the group of $d \times d$ *unitary* matrices, and define $U(d^2)$ similarly.

We now follow Parry [17], and Parry & Pollicott [19], by considering a transfer operator acting on functions taking values in $\mathbb{C}^{d^2}$. Suppose that $\theta \in C^{\mathbf{k}}(X, U(d^2))$ for some $0 < \mathbf{k} \leq \mathbf{s} \leq \mathbf{r}$. As in [17], [19], define the transfer operator L_θ by

$$L_\theta(w)(x) = \sum_{Ty=x} e^{f(y)}\theta(y)w(y). \tag{2}$$

L_θ leaves invariant $L^1(X, \mathbb{C}^{d^2})$, and also $C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$ for $\mathbf{k} \leq \mathbf{s}$.

Since θ is unitary-valued, we see that $|L_\theta^n w(x)|_{d^2} \leq L^n(|w(x)|_{d^2})$ for any $w \in L^1(X, \mathbb{C}^{d^2})$, $x \in X$, and $n \geq 0$. It follows that $\| L_\theta^n \|_{L^1} \leq 1$ for all $n \geq 0$.

Lemma 1. *Suppose $w \in C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$ where $0 < \mathbf{k} = (k, \epsilon) \leq \mathbf{s}$. Then for any $\gamma_0 \in (\gamma, 1)$ there exists $C > 0$ such that for all $n \geq 0$:*

$$\| L_\theta^n w \|_{\mathbf{k}} \leq C \left[\sum_{j=0}^{k} \left(\| D^j w \|_\infty \gamma_0^{nj} \right) + \|D^k w\|_\epsilon \gamma_0^{n|\mathbf{k}|} \right]. \qquad \square$$

Proof. This follows from the fact that there exist $C_1, C_2 > 0$ such that for all $n \geq 0$:

$$\| L_\theta^n w \|_k \leq C_1 \sum_{j=0}^{k} \| D^j w \|_\infty \gamma_0^{nj}$$

and

$$\| D^k(L_\theta^n w) \|_\epsilon \leq C_2 \left[\sum_{j=0}^{k} \left(\| D^j w \|_\infty \gamma_0^{nj} \right) + \|D^k w\|_\epsilon \gamma_0^{n(k+\epsilon)} \right].$$

The proof of the first inequality is rather involved, although the idea behind it is essentially simple. Taking the explicit expression for $L_\theta^n w$ (in terms of a sum over inverse branches of T^n) we use Leibniz' rule to obtain expressions for the various derivatives, up to order k. Systematic use of the chain and product rules allows us to estimate the terms arising in these expressions, and show that higher order derivatives are dominated by powers of the first derivative. A full proof can be found in Appendix A of Jenkinson [3]. Tangerman [29] originally proved this result for the operator L.

The second inequality is a variant of the so-called "basic inequality" (see page 20 of [18]). In addition one uses the techniques described above. □

Note an immediate corollary is that the iterates of L_θ are uniformly bounded in the $C^{\mathbf{k}}$ norm, for $0 < \mathbf{k} \leq \mathbf{s}$.

Now we deduce that the essential spectral radius of the operator L_θ is strictly less than 1. Results of this kind were first proved by Pollicott [21] (cf. also earlier results for the complex Ruelle-Perron-Frobenius operator [20]). For the case of the transfer operator L acting on the space of (real-valued) $C^{\mathbf{k}}$ functions, this has been considered by Tangerman [29] and Ruelle [27].

Proposition 2. *If $0 < \mathbf{k} = (k, \epsilon) \leq \mathbf{s}$ then $C^{\mathbf{k}}(X, \mathbb{C}^{d^2}) = W^1 + W^2$, where*
(i) W^1 and W^2 are both L_θ-invariant,
(ii) W^1 is finite dimensional,
(iii) $L_\theta|_{W^2}$ has spectral radius strictly less than 1.

Proof. For $\gamma_0 \in (\gamma, 1)$, we first show that the part of the spectrum of L_θ lying outside the open disc of radius $\gamma_0^{|\mathbf{k}|}$ consists of a finite number of isolated eigenvalues, each of which has finite multiplicity. To do this we follow Ruelle [27], by defining a sequence of finite rank operators K_n which approximate the iterates L_θ^n. Let $\mathcal{P}^{(n)} = \vee_{i=0}^{n-1} T^{-i}\mathcal{P}$ denote the n^{th} refinement of the Markov partition $\mathcal{P}$. We approximate $w \in C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$ by a local Taylor approximation of order k on each piece of $\mathcal{P}^{(n)}$. This gives a function $w^{(n)}$, say, which is $C^{\mathbf{k}}$ on each piece of $\mathcal{P}^{(n)}$, but is *not* in general $C^{\mathbf{k}}$ on $\mathcal{P}$. The key point is that $K_n(w) := L_\theta^n w^{(n)}$ *is* $C^{\mathbf{k}}$ on $\mathcal{P}$, so we have defined our operator $K_n : C^{\mathbf{k}}(X, \mathbb{C}^{d^2}) \to C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$. We then use Lemma 1, together with some simple estimates (see [27], [22]) to show that there exists $C > 0$ such that, for all $w \in C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$,

$$\|(L_\theta^n - K_n)(w)\|_{\mathbf{k}} \leq C\|w\|_{\mathbf{k}}\gamma_0^{|\mathbf{k}|n}.$$

Now recall that the *essential spectrum* of an operator is obtained by removing from its spectrum all isolated eigenvalues of finite multiplicity. By a formula of Nussbaum (see [16]) we know that the essential spectral radius ϱ of L_θ is $\varrho = \lim_{n\to\infty} \|\!| L_\theta^n \|\!|_{comp}^{1/n}$ where

$$\|\!| L_\theta^n \|\!|_{comp} := \inf \{ \|\!| L_\theta^n - K \|\!|_{\mathbf{k}} : K \text{ is a compact operator } \}.$$

The result follows.

Letting W^1 denote the span of the eigenspaces corresponding to any eigenvalues of modulus one, we then use the Spectral Decomposition Theorem (see [4], page 178) to complete the proof. $\square$

Section 4. Cocycle Rigidity.

Many dynamical problems lead to the study of a cocycle equation (see [5], §2.9) of the form (1), where the functions u, v are known, and of a certain regularity. When the functions are positive real-valued, equation (1) becomes $w(Tx) = u(x)^{-1}v(x)w(x)$, and we have the following result, which is essentially due to Livsic [6], [7]. The proof given below is based on the later method of Parry & Pollicott [19] using transfer operators.

Lemma 3. *Suppose $T : X \to X$ is a piecewise $C^{\mathbf{r}}$, $1 < \mathbf{r} \le \infty$, expanding Markov map. Suppose $\Phi \in C^{\mathbf{k}}(X, \mathbb{R}^+)$ for some $0 < \mathbf{k} = (k, \epsilon) \le \mathbf{s} \le \mathbf{r}$. If the measurable function $W : X \to \mathbb{R}$ satisfies the real-valued cocycle equation*

$$W(Tx) = \Phi(x)W(x) \quad \text{a.e. } (m), \tag{3}$$

then W has a $C^{\mathbf{k}}$ version.

Proof. The proof is in two parts. Firstly, since $\mathbf{k} > 0$, the original work of Livsic implies that W has a $C^{(0,\epsilon)}$ version (see [6], [7], and Walkden [30], Theorem 5.1, as well as the Livsic cocycle rigidity theorem stated in section 1).

Thus we may assume $W \in C^{(0,\epsilon)}(X, \mathbb{R})$, and that (3) holds for *all* $x \in X$.

If $\mathbf{k} < \infty$ then define an operator L_Φ by

$$L_\Phi(u)(x) = \sum_{Ty=x} e^{f(y)}\Phi(y)u(y).$$

Equation (3), together with the fact that $L1 = 1$, means that W is a fixed point of L_Φ.

By the Ruelle-Perron-Frobenius theorem (see Ruelle [26]) we know that L_Φ acting on $C^{\mathbf{k}}$ functions has isolated maximal positive eigenvalue λ, say. Moreover, there exists an eigenvector v of λ such that $\frac{1}{\lambda^n}L_\Phi^n W \to v$ uniformly. That is, $\frac{1}{\lambda^n}W \to v$ uniformly. Thus $\lambda = 1$, so that $W = v$, and hence W is $C^{\mathbf{k}}$.

The case $\mathbf{k} = \infty$ follows by an argument identical to that at the end of the proof of Proposition 4. $\square$

Now we study (1) where u, v are (piecewise) $C^{\mathbf{k}}$ unitary matrix-valued functions, and $T : X \to X$ is our piecewise $C^{\mathbf{r}}$ expanding Markov map. As remarked in the introduction, the non-abelian nature of the problem presents additional difficulties.

The arguments in the remainder of this section follow very closely the scheme of proof given by Parry & Pollicott in [19]. We show that their approach can be adapted to the context of $C^{\mathbf{k}}$ function spaces.

The following proposition should be compared to Theorem 3.1 of Parry [17] (see also Proposition 5 of [19]), where the analogous result is proved for Hölder functions on a subshift of finite type.

Proposition 4. *Suppose the matrix operator L_θ is defined in terms of a piecewise $C^{\mathbf{r}}$, $1 < \mathbf{r} \leq \infty$, expanding Markov map $T : X \to X$, and a function $\theta \in C^{\mathbf{k}}(X, U(d^2))$, where $0 < \mathbf{k} \leq \mathbf{s} \leq \mathbf{r}$. If $w \in L^1(X, \mathbb{C}^{d^2})$ satisfies $L_\theta w = w$ a.e. (m), then w has a $C^{\mathbf{k}}$ version.*

Proof. First assume $\mathbf{k} < \infty$. The fact that $L_\theta w = w$ implies that $\| L_\theta^n \|_{L^1} = 1$ for all $n \geq 0$, since we noted previously that $\| L_\theta^n \|_{L^1} \leq 1$.

Now $C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$ is L^1-dense in $L^1(X, \mathbb{C}^{d^2})$, so choose a sequence $w_j \in C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$ with $w_j \to w$ in L^1. By Proposition 2 we can write $w_j = w_j^1 + w_j^2$, where $w_j^1 \in W^1$ and $w_j^2 \in W^2$.

As observed after Proposition 1, the sequence of iterates $L_\theta^n w_j^1$ is $C^{\mathbf{k}}$-bounded. But the L_θ-invariant subspace W^1 is finite dimensional, so in fact the sequence is contained in a *compact* subset of W^1, and hence has a $C^{\mathbf{k}}$ limit point $w_j^\star \in W^1$

Since the spectral radius of $L_\theta|_{W^2}$ is strictly less than one, we have $\lim_{n\to\infty} L_\theta^n w_j^2 = 0$, so that $w_j^\star$ is also a $C^{\mathbf{k}}$ limit point for the sequence $L_\theta^n w_j$. Thus there is a subsequence $(a_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} L_\theta^{a_n} w_j = w_j^\star$ in the $C^{\mathbf{k}}$ norm.

Now consider the sequence $w_j^\star$. Since $\| L_\theta^n \|_{L^1} \leq 1$ for all $n \geq 0$, we have

$$\begin{aligned} \int |w - w_j^\star|_{d^2}\, dm &= \lim_{n\to\infty} \int |w - w_j^\star|_{d^2}\, dm \\ &\leq \lim_{n\to\infty} \left[\int |L^{a_n}(w - w_j)|_{d^2}\, dm + \int |L^{a_n}(w_j - w_j^*)|_{d^2}\, dm \right] \\ &= \int |w - w_j|_{d^2}\, dm + 0. \end{aligned} \tag{4}$$

Thus $w_j^\star \to w$ in L^1. But each $w_j^\star \in W^1$, and all norms are equivalent in the finite dimensional space W^1, so $w_j^\star$ is convergent in the $C^{\mathbf{k}}$ norm. So the pointwise limit $w'(x) := \lim_{j\to\infty} w_j^\star(x)$ exists for all $x \in X$, and since w' is the $C^{\mathbf{k}}$ limit of a sequence of $C^{\mathbf{k}}$ functions, then $w' \in C^{\mathbf{k}}(X, \mathbb{C}^{d^2})$. Clearly $w = w'$ a.e., so w' is the required $C^{\mathbf{k}}$ version of w.

Now suppose $\mathbf{k} = \infty$. The above argument means that for all $l \geq 1$ there exists $w' \in C^{(l,0)}(X, M(d))$ such that $w = w'$ almost everywhere. Clearly w' is independent of l, so $w' \in C^\infty(X, M(d))$. $\square$

To relate Proposition 4 to the matrix cocycle equation (1), we first recall some standard facts about matrices.

Lemma 5.

(i) If $A, B \in M(d)$ then $< A, B >_{d^2} = Trace(AB^\star)$,

(ii) $Trace\,(CD) = Trace\,(DC)$ for all $C, D \in M(d)$,

(iii) If $A \in M(d^2)$, then $A \in U(d^2)$ if and only if $< Ax, Ay >_{d^2} = < x, y >_{d^2}$ for all $x, y \in \mathbb{C}^{d^2}$.

In the proof of the next theorem we employ a trick due to Parry & Pollicott (see Theorem 1 in [19]). This trick, which illuminates the whole approach so far, is to explicitly define the function θ in terms of the $U(d)$-valued functions u, v of (1), and to then identify $d \times d$ matrices as vectors in $\mathbb{C}^{d^2}$, so as to interpret θ as a $U(d^2)$-valued function.

Theorem A. *Suppose $T : X \to X$ is a piecewise $C^{\mathbf{r}}$, $1 < \mathbf{r} \leq \infty$, expanding Markov map, with $u, v \in C^{\mathbf{k}}(X, U(d))$ for some $0 < \mathbf{k} \leq \mathbf{s} \leq \mathbf{r}$. If $w \in L^1(X, M(d))$ satisfies the matrix cocycle equation*

$$w(Tx)u(x) = v(x)w(x) \quad \text{a.e. } (m), \tag{5}$$

then w has a $C^{\mathbf{k}}$ version.

Proof. Since $u(x)$ is unitary, equation (5) becomes

$$w(Tx) = v(x)w(x)u(x)^\star \quad \text{a.e.} \tag{6}$$

For each $x \in X$, define the linear map

$$\begin{aligned} \theta(x) &: M(d) \to M(d) \\ &: A \mapsto v(x)Au(x)^\star, \end{aligned}$$

and note that

$$\begin{aligned} < \theta(x)(A), \theta(x)(B) >_{d^2} &= < v(x)Au(x)^\star, v(x)Bu(x)^\star >_{d^2} \\ &= \text{Trace}\,[v(x)Au(x)^\star(v(x)Bu(x)^\star)^\star] \quad \text{by Lemma 5 (i)} \\ &= \text{Trace}\,[v(x)Au(x)^\star u(x)B^\star v(x)^\star] \\ &= \text{Trace}\,[v(x)AB^\star v(x)^\star] \quad \text{since } u(x) \text{ is unitary} \\ &= \text{Trace}\,[AB^\star] \quad \text{by Lemma 5 (ii)} \\ &= < A, B >_{d^2} \quad . \end{aligned}$$

Thus $\theta(x) : M(d) \to M(d)$ is a unitary map, by Lemma 5 (iii), and identifying $M(d)$ with $\mathbb{C}^{d^2}$, we can think of $\theta(x)$ as a $d^2 \times d^2$ unitary *matrix* acting on $\mathbb{C}^{d^2}$. By the same identification we can think of w as belonging to $L^1(X, \mathbb{C}^{d^2})$.

So (6) is equivalent to the equation $w(Tx) = \theta(x)w(x)$ a.e., which in turn is equivalent to $L_\theta w = w$ a.e. (one direction is easy, for the other see [17], p. 398). Since u, v are $C^{\mathbf{k}}$ we have that $\theta \in C^{\mathbf{k}}(X, U(d^2))$, and the result follows by Proposition 4. $\square$

Corollary A.1. *Suppose $T : X \to X$ is a piecewise $C^{\mathbf{r}}$, $1 < \mathbf{r} \leq \infty$, expanding Markov map, with $u, v \in C^{\mathbf{k}}(X, G)$ for some $0 < \mathbf{k} \leq \mathbf{s} \leq \mathbf{r}$, and G a compact Lie group. If the measurable function $w : X \to G$ satisfies*

$$w(Tx)u(x) = v(x)w(x) \quad \text{a.e. } (m),$$

then w has a $C^{\mathbf{k}}$ version.

Proof. The measurable function w is automatically L^1, and by representation theory we can choose d such that G is a subgroup of $U(d)$, so that the result follows from Theorem A. $\square$

Section 5. Conjugacy Rigidity.

We now apply our cocycle rigidity results to the problem of conjugacy rigidity. For $i = 1, 2$, suppose $T_i : X_i \to X_i$ are piecewise $C^{\mathbf{r}}$, $1 < \mathbf{r} = (r, \delta) \leq \infty$, expanding Markov maps with respect to partitions $\mathcal{P}_i$ of spaces X_i. Here we assume the X_i both lie in $\mathbb{R}^d$ (i.e. we set $p = d$), so that the derivatives of the T_i will be $d \times d$ matrices, as were the functions u, v in Theorem A. Henceforth, our reference equilibrium state m will be the unique T_1-invariant probability measure which is absolutely continuous with respect to Lebesgue measure. The corresponding smooth potential is $f(x) = -\log |\det D_x T_1|$, so that $\mathbf{s} = \mathbf{r} - 1$.

A homeomorphism $h : X_1 \to X_2$ satisfying $hT_1 = T_2 h$ is called a *topological conjugacy* between T_1 and T_2. The fact that T_1, T_2 are expanding implies that h is piecewise $C^{(0,\alpha)}$, for some $0 < \alpha < 1$ (the proof for Anosov diffeomorphisms on page 599 of [5] is easily adapted). However in general h will *not* be any more regular than Hölder, and in particular need not be Lipschitz. Consider, for example, the piecewise linear family

$$T_a(x) = \begin{cases} \frac{x}{a} & \text{if } x \in [0, a) \\ \frac{x-a}{1-a} & \text{if } x \in [a, 1) \end{cases}$$

where $a \in (0, 1)$. Each T_a is topologically conjugate to the doubling map $T_{1/2}$, with conjugacy h mapping $(0, a)$ to $(0, 1/2)$, and $(a, 1)$ to $(1/2, 1)$. Lebesgue measure l is invariant for both maps. Thus the push-forward measure $h_*(l)$ is invariant for $T_{1/2}$, and corresponds to Bernoulli $(a, 1 - a)$ measure. For $a \neq 1/2$ the set of $h_*(l)$-generic points for $T_{1/2}$ has zero Lebesgue measure, but is mapped by h to the set of l-generic points for $T_{1/2}$, which has full Lebesgue measure. The presence of such irregular conjugacies h motivates the following definition.

A topological conjugacy h between T_1 and T_2 is called *absolutely continuous* if the set $\{x \in X_1 : h \text{ is differentiable at } x \text{ and } h^{-1} \text{ is differentiable at } h(x)\}$ has full Lebesgue measure, and all partial derivatives of h and h^{-1} are L^2 functions.

Any bi-Lipschitz h is absolutely continuous, by Rademacher's Theorem (see Morgan [11], p. 21), as is any quasi-conformal homeomorphism in dimension two or more (see).

In the sequel we make the further assumption that T_1, T_2 are *(piecewise) conformal*, that is, the derivative at each point is a non-zero multiple of an isometry. So we have $a_i : X_i \to \mathbb{R}$ and $U_i : X_i \to SO(d, \mathbb{R})$ such that $D_x T_i = a_i(x) U_i(x)$.

The result we need for our application to Mostow rigidity is that any absolutely continuous conjugacy between T_1, T_2 must necessarily be smooth. C^1 smoothness is enough for our purposes (see), but our methods in fact show the conjugacy is as $C^{\mathbf{r}}$ smooth as the maps T_1, T_2.

Theorem B is also true (by the same proof) if the functions U_i above are assumed to take values in some compact subgroup H of $SL(d, \mathbb{R})$. The class of corresponding maps T_i is not significantly wider, however, as any such H can be conjugated into a subgroup of $SO(d, \mathbb{R})$ by an element of $GL(d, \mathbb{R})$, and the T_i would then both be conjugate to conformal maps via the same element of $GL(d, \mathbb{R})$.

The general approach to analysing the regularity of conjugacies by reducing to Livsic's cocycle equation was pioneered by de la Llave, Marco & Moriyón [9]. (Previous methods had relied on geometrically less natural generating functions). The "bootstrap of regularity" technique used in the proof of Theorem B, to deduce successively higher regularity of the conjugacy by repeated application of Theorem A, appears in de la Llave [8].

Theorem B. *For $1 < \mathbf{r} = (r, \delta) \leq \infty$, let $T_i : X_i \to X_i$ be piecewise $C^{\mathbf{r}}$ conformal expanding Markov maps. Any absolutely continuous conjugacy between T_1 and T_2 is itself piecewise $C^{\mathbf{r}}$.*

Proof. It suffices to prove the result for $\mathbf{r} < \infty$, as the C^∞ case follows immediately. If $hT_1 = T_2 h$ we will show that h is $C^{\mathbf{r}}$. An analogous argument shows that h^{-1} is $C^{\mathbf{r}}$.

Step 1. Reduce the problem to two cocycle equations

Differentiating the conjugacy equation (almost everywhere) we obtain

$$D_{T_1(x)} h . D_x T_1 = D_{h(x)} T_2 . D_x h \quad \text{a.e.} \tag{7}$$

The T_i are conformal so we write $D_x T_i = a_i(x) U_i(x)$, and (7) becomes

$$D_{T_1(x)} h . a_1(x) U_1(x) = a_2(h(x)) U_2(h(x)) . D_x h \quad \text{a.e.} \tag{8}$$

We want to split equation (8) into two equations, one involving real-valued functions, the other involving matrix-valued functions.

Since $U_1(x), U_2(h(x)) \in SO(d, \mathbb{R})$ for all $x \in X_1$, taking the determinant of both sides of (8) gives

$$\det[D_{T_1(x)} h] a_1(x)^d = a_2(h(x))^d \det[D_x h] \quad \text{a.e.} \tag{9}$$

If d is *odd* we can straightaway take unique real d^{th} roots of both sides of (9) to obtain

$$(\det[D_{T_1(x)}h])^{1/d} a_1(x) = a_2(h(x))(\det[D_x h])^{1/d} \quad \text{a.e.} \tag{10}$$

Substituting (12) into the matrix equation (8) gives

$$((\det[D_{T_1(x)}h])^{-1/d} D_{T_1(x)}h).U_1(x) = U_2(h(x)).((\det[D_x h])^{-1/d} D_x h) \quad \text{a.e.} \tag{11}$$

If d is *even* we may assume that a_1, a_2 are both positive-valued, since in this case a matrix $U \in SO(d,\mathbb{R})$ if and only if $-U \in SO(d,\mathbb{R})$. Moreover, equation (9) implies that $\det[D_{T_1(x)}h]$ and $\det[D_x h]$ have the same sign. Taking absolute values and positive d^{th} roots of both sides of (9) gives

$$|\det[D_{T_1(x)}h]|^{1/d} a_1(x) = a_2(h(x))|\det[D_x h]|^{1/d} \quad \text{a.e.} \tag{12}$$

and substituting (12) into the matrix equation (8) gives

$$(|\det[D_{T_1(x)}h]|^{-1/d} D_{T_1(x)}h).U_1(x) = U_2(h(x)).(|\det[D_x h]|^{-1/d} D_x h) \quad \text{a.e.} \tag{13}$$

To clarify equations (11) and (13) we define the function $w : X_1 \to M(d)$ by

$$w(x) = \begin{cases} (\det[D_x h])^{-1/d} D_x h = (\det[D_{h(x)}h^{-1}])^{1/d} D_x h & \text{if } d \text{ is odd} \\ |\det[D_x h]|^{-1/d} D_x h = |\det[D_{h(x)}h^{-1}]|^{1/d} D_x h & \text{if } d \text{ is even,} \end{cases}$$

and note the absolute continuity of h means w is defined almost everywhere and measurable. Now $\det[D_{h(x)}h^{-1}]$ is the sum of $d!$ products, where each product has d factors, and each factor is a partial derivative of h^{-1}, and hence is L^2. By Hölder's inequality we deduce that each of the products is $L^{2/d}$, thus $\det[D_{h(x)}h^{-1}]$ is also $L^{2/d}$. So both $(\det[D_{h(x)}h^{-1}])^{1/d}$ and $|\det[D_{h(x)}h^{-1}]|^{1/d}$ are L^2. Now $D_x h$ is also an L^2 function of x, so applying Hölder's inequality gives $w \in L^1(X_1, M(d))$.

Then equations (11) and (13) both become

$$w(T_1(x)).U_1(x) = U_2(h(x)).w(x) \quad \text{a.e.,} \tag{14}$$

where U_1, U_2 take values in $SO(d,\mathbb{R})$, and the solution w is integrable.

Returning to the real-valued equations (10) and (12), we define $W : X_1 \to \mathbb{R}$ by

$$W(x) = \begin{cases} (\det D_x h)^{1/d} & \text{if } d \text{ is odd} \\ |\det D_x h|^{1/d} & \text{if } d \text{ is even} \end{cases}$$

on the set of full measure where these expressions make sense, and then derive

$$W(T_1(x)) = \frac{a_2(h(x))}{a_1(x)} W(x) \quad \text{a.e.} \tag{15}$$

This is a real-valued cocycle equation whose solution W is measurable.

Note that $D_x h = W(x)w(x)$ for almost all $x \in X_1$.

Step 2. Prove that h is $C^{(0,\delta)}$.

If $\alpha \geq \delta$ then this is immediate. If $\alpha < \delta$ we will show h is $C^{(1,\alpha)}$, and hence $C^{(0,\delta)}$.

Define $u, v : X_1 \to SO(d, \mathbb{R})$ by $u(x) = U_1(x)$ and $v(x) = U_2(h(x))$. (14) gives

$$w(T_1(x))u(x) = v(x)w(x) \quad \text{a.e.} \tag{16}$$

where $w \in L^1(X_1, M(d))$. Now since h is $C^{(0,\alpha)}$, we see u, v are also both $C^{(0,\alpha)}$, so by Theorem A (with $\mathbf{k} = (0, \alpha)$) we deduce w has a $C^{(0,\alpha)}$ version w'.

Applying essentially the same argument to the real-valued cocycle equation (15) we deduce (using Lemma 3 with $\mathbf{k} = (0, \alpha)$) that W has a $C^{(0,\alpha)}$ version W'.

So $D_x h = W(x)w(x) = W'(x)w'(x)$ for almost all $x \in X_1$, but since Dh is a derivative of a continuous function we see that in fact this equality holds for all $x \in X_1$. So Dh is $C^{(0,\alpha)}$, thus h is $C^{(1,\alpha)}$, and in particular is $C^{(0,\delta)}$.

Step 3. Prove that h is $C^{(r,\delta)}$.

The proof is by induction. Our inductive hypothesis is that h is $C^{(k,\delta)}$ for some $0 \leq k \leq r - 1$, and we have just seen that this is true for $k = 0$. We will show this implies h is $C^{(k+1,\delta)}$.

The inductive hypothesis means that the functions u, v in the matrix-valued cocycle equation (16) are both $C^{(k,\delta)}$ (recall that v was defined in terms of h). By Theorem A, and since we now know w is continuous, we deduce w is $C^{(k,\delta)}$. A similar argument applied to equation (15), and using Lemma 3, shows W is $C^{(k,\delta)}$.

Thus $Dh = Ww$ is $C^{(k,\delta)}$, so that h is $C^{(k+1,\delta)}$, as required. $\square$

Section 6. Bowen-Series Maps and Mostow Rigidity.

Let $\mathbb{H}$ be the Poincaré ball model of hyperbolic 3-space. A Kleinian group Γ (i.e. a discrete subgroup of isometries of $\mathbb{H}$) is *geometrically finite* if it has a fundamental polyhedron $\mathcal{R}$ with only finitely many faces $F_1, \ldots, F_r$ (see [10]). In this case each F_i is part of an isometric sphere of some $g_i \in \Gamma$, and the elements $g_1, \ldots, g_r$ form a generating set for Γ. Each face F_i is mapped by g_i to another face F_j. We say that $\mathcal{R}$ is *even-cornered* if each isometric sphere of a generator g_i lies completely in $\Gamma(\partial\mathcal{R})$, where $\partial\mathcal{R}$ denotes the boundary of $\mathcal{R}$.

The action of a geometrically finite cocompact Kleinian group with an even-cornered fundamental polyhedron can be modelled by an expanding Markov map T on $S^2 = \partial\mathbb{H}$. This *Bowen-Series map* can be defined by

$$T(x) = g_i(x) \quad \text{if } x \in A(g_i) \setminus \bigcup_{j=1}^{i-1} A(g_j),$$

where $A(g_i)$ denotes the intersection with S^2 of the isometric ball of the generator g_i. The ordering $g_1 < g_2 < \ldots < g_r$ on the generating set is chosen arbitrarily, so there are several choices of Bowen-Series map. For each $x \in S^2$, the key property is that $y \in \Gamma(x)$ if and only if there exist $m, n \geq 0$ with $T^m(x) = T^n(y)$. This construction is due originally to Bowen & Series [2] in the context of Fuchsian groups, while the generalisation to Kleinian groups is due to Rocha [24] (see also [1]).

Now suppose M_1, M_2 are compact connected orientable hyperbolic 3-manifolds with the same homotopy type. By Mostow's rigidity theorem these manifolds are isometric.

In the classical proof of Mostow rigidity (see [12]) one first shows there is some absolutely continuous homeomorphism $h : S^2 \to S^2$, which is conformal almost everywhere, such that $h(\Gamma_1(x)) = \Gamma_2(h(x))$ for all $x \in S^2$, where Γ_i denotes the fundamental group of M_i. A significant step is then to show that h is *everywhere conformal*.

Now if the Γ_i admit Bowen-Series maps T_i, then we can prove the conformality of h as follows. The equation $h\Gamma_1 = \Gamma_2 h$ becomes $hT_1 = T_2 h$, so that h is an absolutely continuous conjugacy between T_1, T_2, which in particular are piecewise $C^{(1,\delta)}$ conformal expanding Markov maps. Applying Theorem B we deduce that h is a piecewise $C^{(1,\delta)}$ diffeomorphism, and in particular is C^1. The possible non-smoothness on the partition boundaries is an artefact of our construction, since the *same* h conjugates *any* choice of Bowen-Series maps T_1, T_2. In particular, the flexibility in ordering the generators of Γ_1, Γ_2 in the definition of T_1, T_2 means the T_i can be chosen to be smooth over any specified boundary component of the Markov partition (considered as a covering of S^2), so we deduce that h is smooth here as well.

Thus h is a C^1 diffeomorphism of S^2. In particular, the derivative Dh is continuous. But conformality almost everywhere with respect to a fully supported measure implies conformality on a dense set, so that the continuity of Dh implies that h is in fact conformal *everywhere*.

Acknowledgments. I would like to thank M. Pollicott for many valuable conversations regarding this work. I am also grateful to J. Anderson, V. Baladi, A. Rocha, H-H. Rugh and C. Walkden for various conversations and correspondence.

References

[**1**] J. Anderson and A. Rocha, Analyticity of Hausdorff dimension of limit sets of Kleinian groups, *Ann. Acad. Sci. Fenn. Math.*, **22** (1997), 349–364.

[**2**] R. Bowen and C. Series, Markov maps associated to Fuchsian groups, *I.H.E.S. Publ. Math.*, **50** (1979), 401–418.

[**3**] O. Jenkinson, Conjugacy rigidity, cohomological triviality, and barycentres of invariant measures, *Ph.D. Thesis*, Warwick University.

[**4**] T. Kato, *Perturbation Theory*, Springer-Verlag, Berlin, 1976.

[**5**] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[**6**] A. Livsic, Homology properties of Y-systems, *Math. Zametki*, **10** (1971), 758–763.

[**7**] A. Livsic, Cohomology of dynamical systems, *Math. USSR Izvestija*, **6** (1972), 1278–1301.

[**8**] R. de la Llave, Invariants for smooth conjugacy of hyperbolic dynamical systems II, *Comm. Math. Phys.*, **109** (1987), 369–378.

[**9**] R. de la Llave, J. M. Marco and R. Moriyón, Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation, *Ann. of Math.*, **123** (1986), 537–611.

[**10**] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin, 1988.

[**11**] F. Morgan, *Geometric Measure Theory*, Academic Press, San Diego, 1988.

[**12**] G. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, *I.H.E.S. Publ. Math.*, **34** (1968), 53–104.

[**13**] M. Nicol and M. Pollicott, Measurable cocycle rigidity for some non-compact groups, *Bull. London Math. Soc.*, **31** (1999), 592–600.

[**14**] V. Nitica and A. Torok, Regularity results for the solutions of the Livsic cohomology equation with values in diffeomorphism groups, *Ergod. Th. & Dyn. Sys.*, **16** (1996), 325–333.

[**15**] V. Nitica and A. Torok, Regularity of the transfer map for cohomologous cocycles, *Ergod. Th. & Dyn. Sys.*, **18** (1998), 1187–1209.

[**16**] R. Nussbaum, The radius of the essential spectrum, *Duke Math. J.*, **37** (1970), 473–478.

[**17**] W. Parry, Skew-products of shifts with a compact Lie group, *J. London Math. Soc. (2)* **56** (1997), 395–404.

[**18**] W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque 187–188, 1990.

[**19**] W. Parry and M. Pollicott, The Livsic cocycle equation for compact Lie group extensions of hyperbolic systems, *J. London Math. Soc. (2)* **56** (1997), 405–416.

[**20**] M. Pollicott, A complex Ruelle-Perron-Frobenius theorem and two counterexamples, *Ergod. Th. Dyn. Sys.*, **4** (1984), 135–146.

[**21**] M. Pollicott, Meromorphic extensions of generalised zeta functions, *Invent. Math.* **85** (1986), 147–164.

[**22**] M. Pollicott, On the Ruelle-Tangerman theorem for zeta functions, *Proceedings of the European Conference on Iteration Theory*, Lisbon, 1991, 201–209.

[**23**] M. Pollicott and C. Walkden, Livsic theorems for connected Lie groups, *Trans. Amer. Math. Soc.*, to appear.

[**24**] A. Rocha, Symbolic dynamics for Kleinian groups, *Ph.D. Thesis*, Warwick University (1994).

[**25**] A. Rocha, Meromorphic extension of the Selberg zeta function for Kleinian groups via thermodynamic formalism, *Math. Proc. Cambridge Philos. Soc.*, **119** (1996), 179–190.

[**26**] D. Ruelle, *Thermodynamic Formalism*, Encyclopaedia of Mathematics and its Applications, vol. 5, Addison-Wesley, 1978.

[**27**] D. Ruelle, An extension of the theory of Fredholm determinants, *I.H.E.S. Publ. Math.*, **72** (1990), 175–193.

[**28**] K. Schmidt, Remarks on Livsic' theory for nonabelian cocycles, Ergod. Th. & Dyn. Sys., **19** (1999), 703–721.

[**29**] F. Tangerman, Meromorphic continuation of Ruelle zeta functions, *Ph.D. Thesis*, Boston University (1986).

[**30**] C. Walkden, Livsic Theorems for Hyperbolic Flows, *Trans. Amer. Math. Soc.*, **352** (2000), 1299-1313.

[**31**] P. Walters, Ruelle's operator theorem and g-measures, *Trans. Amer. Math. Soc.*, **214** (1975), 375–387.