ERGODIC PROPERTIES OF THE BOLYAI-RÉNYI EXPANSION

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Abstract. We consider the ergodic properties of the Bolyai-Rényi expansion for real numbers introduced in [Bol], [Ren]. We compute the entropy of the absolutely continuous invariant probability measure μ of the associated dynamical system, resolving a conjecture of Bosma, Dajani and Kraaikamp [BDK]. We also compute the frequencies with which the various digits occur in the expansions of Lebesgue almost every number. Finally, we calculate the rate of decay of correlations with respect to μ.

1. Introduction

In his influential 1957 paper [Ren], Rényi considered the representation of real numbers $x \in [0,1]$ in terms of various "f-expansions" of the form

$$\begin{aligned} x &= f\left(a_1 + f\left(a_2 + f\left(a_3 + \dots\right)\right)\right) \\ &= \lim_{n\to\infty} f\left(a_1 + f\left(a_2 + \dots f\left(a_n\right)\dots\right)\right) \end{aligned}$$

for certain integer digits $a_i \geq 0$. For example, the function $f(x) = 1/x$ gives the familiar continued fraction expansion

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

The choice $f(x) = x/\beta$, for $\beta > 1$ gives the β-expansion of a number, which is a natural generalisation of its decimal expansion. Rényi also considered the choice $f(x) = \sqrt{1+x} - 1$, leading to the expansion

$$x = -1 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \dots}}},$$

where each $a_i \in \{0,1,2\}$.

This expansion had been originally introduced by Bolyai in 1832 (see the 1897 volume [Bol]) to approximate roots of certain equations. It has also recently been considered by Bosma, Dajani & Kraaikamp [BDK].

We would like to thank Cor Kraaikamp for suggesting this problem to us.

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The Bolyai-Rényi expansion of a number x is given (see section 2 for details) by iterating the following map T of the unit interval,

$$T(x) = \begin{cases} x^2 + 2x & \text{if } 0 \le x < \sqrt{2} - 1 \\ x^2 + 2x - 1 & \text{if } \sqrt{2} - 1 \le x < \sqrt{3} - 1 \\ x^2 + 2x - 2 & \text{if } \sqrt{3} - 1 \le x \le 1, \end{cases}$$

and the statistical properties of these expansions are given by the ergodic properties of T. The study of these ergodic properties has been hindered by the fact that there is no known formula for the absolutely continuous T-invariant probability measure μ. The purpose of this note is to provide very accurate *approximate* ergodic properties of the Bolyai-Rényi expansion with respect to μ.

Firstly, we compute the corresponding metric entropy $h(\mu)$, which is also equal to the Lyapunov exponent of μ. The entropy reflects the average "quality" of the expansion (see Proposition 1 and Corollary 2). We derive (Theorem 1) the estimate $h(\mu) = 1.05631307402$, and prove that this is within 1.3×10^{-7} of the true entropy. We also give an *empirical* estimate $h(\mu) = 1.056313074$. By an empirical estimate we mean that we are confident this value is correct up to the accuracy given, and that this is strongly supported by numerical evidence (see below for more details of our criterion).

Secondly, we compute the frequencies of the digits 0, 1 and 2 occurring in almost every expansion. In Theorem 2 we prove that

$$\begin{aligned} freq(0) &= 0.46407962943 \pm 2.9 \times 10^{-8} \\ freq(1) &= 0.3044190449 \pm 1.6 \times 10^{-8} \\ freq(2) &= 0.2315013256 \pm 1.1 \times 10^{-8} \end{aligned}$$

where $freq(i) = \lim_{N \to +\infty} \frac{1}{N} \#\{1 \le k \le N : a_k = i\}$. From this we derive (Corollary 3) the asymptotic arithmetic mean of the digits.

Thirdly, it is well known that the *correlation function* $c_n(f,g) = \int f \cdot T^n g \, d\mu - \int f \, d\mu \int g \, d\mu$ of two analytic functions f, g converges to zero at some exponential rate $O(\alpha^n)$, $0 < \alpha < 1$, which can be chosen independent of f, g. We estimate (Theorem 3) this rate of decay of correlations to be $\alpha = 0.40582$, and prove rigorously that this is within 0.000076 of the true value.

In our calculations we exploit the analyticity of T, and combine this with some ideas from thermodynamic formalism. To compute the entropy we use the Rohlin-Pesin formula $h(\mu) = \int \log |T'| d\mu$. By computing the *periodic points* of T we derive an expression for $h(\mu)$ as the ratio of two rapidly converging series expansions. The approximations to $h(\mu)$ are given by working with the corresponding order-n truncated series. We obtain our *empirical* estimate by choosing n as large as is computationally feasible, then computing both the order-n and the order-$(n-1)$ truncations. We then compare these estimates, and retain those digits which agree. The (non-rigorous) justification for this is an empirical observation of the rapid convergence rate (see Table 1). To obtain our *rigorous* estimate, we combine an *exact* calculation of the first 12 terms in each series, with a rigorous *estimate* on the tail of the series from the 13^{th} term onwards. To obtain this tail estimate we give explicit bounds on the rate of decay of terms in the series expansions. The computation of digit frequencies is similar, though here we integrate characteristic functions rather than $\log |T'|$.

It is well known that the rate of correlation decay is by the second largest eigenvalue of the Perron-Frobenius operator $\mathcal{L}$ (defined in section 3). When $\mathcal{L}$ acts on a suitable analytic function space, its eigenvalues are given by the reciprocals of the zeros z of its Fredholm determinant. This determinant is an entire function, so by computing the zeros of its polynomial truncations we obtain rapid approximations to the true rate of correlation decay.

The article is organised as follows. In Section 2 we recall some familiar properties of f-expansions, and discuss a few examples. In Section 3 we specialise to the Bolyai-Rényi map, considering its extension to certain discs in the complex plane, and introducing a family of transfer operators. In Section 4 we show that these operators are nuclear, and obtain explicit bounds on the coefficients in the power series expansions of their Fredholm determinants. Sections 5, 6 and 7 are devoted, respectively, to the proofs of the results on the entropy, digit frequencies, and rate of correlation decay.

2. f-EXPANSIONS

To each of the f-expansions described in the introduction, we can associate an expanding map T of the unit interval I, defined by $T(x) = \{f^{-1}(x)\}$, where $\{\cdot\}$ denotes the fractional part.

Lemma 1. *Such expanding maps always possess an invariant probability measure μ, absolutely continuous with respect to Lebesgue measure λ* [LaY]. *This absolutely continuous invariant measure is unique.*

Let $h(\mu)$ denote the entropy of μ. Let $a_k = a_k(x)$ denote the k^{th} digit in the f-expansion of x, and denote $I_i = \{x : a_1(x) = i\}$.

Proposition 1.
(1) *$freq(i) := \lim_{N\to+\infty} \frac{1}{N}\#\{1 \le k \le N : a_k(x) = i\} = \mu(I_i)$ for almost all points x,*
(2) *If we denote $x_N = f(a_1 + f(a_2 + \ldots + f(a_N)\ldots))$, $N \ge 1$, then for any $\varepsilon > 0$, and almost all points x, we have that $|x - x_N| = O(e^{-N(h(\mu)-\varepsilon)})$.*

Proof. Part (1) is a consequence of the Birkhoff Ergodic Theorem, while part (2) follows from the Shannon-McMillan-Breiman theorem, (see for example [Wal]).

Example 1. In the case of continued fractions we have the Gauss map $T(x) = \{1/x\}$, a Markov map with infinitely many branches. The invariant density $d\mu/d\lambda(x) = (\log 2(1+x))^{-1}$ is known explicitly, and this allows an exact calculation of the corresponding metric entropy as $\frac{\pi^2}{6\log 2}$, see [Sch].

Example 2. The β-transformation $T(x) = \{\beta x\}$ is in general not a Markov map (see [Bl]), though the function $d\mu/d\lambda(x) = \sum_{n=0}^{\infty} \beta^{-n}\chi_{\{z:z<T^n(1)\}}(x)$ gives an exact formula for the (non-normalised) invariant density of μ (see [Pa]). The entropy is simply $\log\beta$, by the Rohlin-Pesin formula [Ro], [Pe].

The symbolic dynamics of the Bolyai-Rényi map $T(x) = \{(x+1)^2 - 1\}$ is particularly simple, a full shift on a three letter alphabet. However, despite the *existence* of an absolutely continuous invariant measure μ, an exact formula for its density function is not known. Consequently, other statistical information such as the frequency of digits, the metric entropy, the rate of correlation decay, etc. are also unknown.

In the Bolyai-Rényi case, T is an expanding Markov map with respect to the partition consisting of intervals $I_0 = [0, \sqrt{2}-1]$, $I_1 = [\sqrt{2}-1, \sqrt{3}-1]$, $I_2 = [\sqrt{3}-1, 1]$. The corresponding inverse branches $T_j : I \to I_j$ are given by

$$T_j(x) = \sqrt{j+1+x} - 1.$$

Given $x \in [0,1]$, the sequence of digits $a_n = a_n(x)$ is generated by setting $r_0 = x$, then defining recursively

$$r_{n+1} = \{(1+r_n)^2 - 1\} \qquad a_{n+1} = [(1+r_n)^2 - 1].$$

Equivalently, for $x \in [0,1]$, we have $r_n = T^n x$, and a_n is such that $T^{n-1}x \in I_{a_n}$.

The sequence of digits is unique unless some iterate of x lies on the endpoint of a piece of the Markov partition. The set of such points is countable, and in particular of zero Lebesgue measure, so we freely ignore them in the sequel.

3. Preliminaries

Our analysis will involve the periodic points of the Bolyai-Rényi map T. The two endpoints 0 and 1 are fixed by T, as is the golden ratio

$$\gamma = -1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

For a given string $\underline{j} = j_1 \ldots j_n \in \{0,1,2\}^n$ of length $|\underline{j}| = n$, we define the contraction $T_{\underline{j}}$ to be the composition $T_{j_1} \cdot \ldots \cdot T_{j_n}$. The algebraic number $x_{\underline{j}}$, defined as the fixed point of $T_{\underline{j}}$, is a period-n point under T, and all period-n points arise in this way.

For a general closed disc $D \subset \mathbb{C}$, let $A_\infty(D)$ denote the Banach space of complex-valued functions which are continuous on D and holomorphic on $int(D)$, equipped with the supremum norm.

Now let us fix the disc D to be centred at $1/2$, of radius $\sqrt{3}/2 \approx 0.866$. Each branch T_j has an analytic extension to D, which we also denote by T_j. Its derivative extends to

$$T_j'(z) = \frac{1}{2\sqrt{z+j+1}} = \frac{1}{2(T_j z + 1)}.$$

Define the composition-type operators $\mathcal{L}_j w(z) = w(T_j z) T_j'(z)$, and then the *Perron-Frobenius operator* $\mathcal{L} : A_\infty(D) \to A_\infty(D)$ by

$$\mathcal{L}w(z) = \sum_{j=0}^{2} \mathcal{L}_j w(z) = \sum_{j=0}^{2} \frac{w(T_j z)}{2\sqrt{z+j+1}} = \sum_{j=0}^{2} \frac{w(T_j z)}{2(T_j z + 1)}. \tag{3.1}$$

We will be interested in integrating various analytic functions g with respect to the measure μ. For example, the choice $g(x) = \log|T'(x)| = \log(2x+2)$ (see Section 5) gives the entropy $h(\mu)$.

Given any function $g \in A_\infty(D)$, for all $t \in \mathbb{C}$ we also have $e^{tg \cdot T_j} \in A_\infty(D)$, for each $j = 0,1,2$, so define *transfer operators* $\mathcal{L}_{g,t}$ on $A_\infty(D)$ by

$$\mathcal{L}_{g,t} w(z) = \sum_{j=0}^{2} \frac{e^{tg(T_j z)} w(T_j z)}{2\sqrt{z+j+1}} = \sum_{j=0}^{2} \frac{e^{tg(T_j z)} w(T_j z)}{2(T_j z + 1)}. \tag{3.2}$$

The closed disc D', centred at $1/2$ and radius $\frac{3}{2}(\sqrt{3}-1) \approx 1.098$, contains the smaller disc D. The disc D' is also contained in each of the (local) inverse images $T_j^{-1}D$. In other words,

$$T_1(D') \cup T_2(D') \cup T_3(D') \subset D.$$

Moreover, each of the derivatives T_j' extend analytically to the larger disc D'.

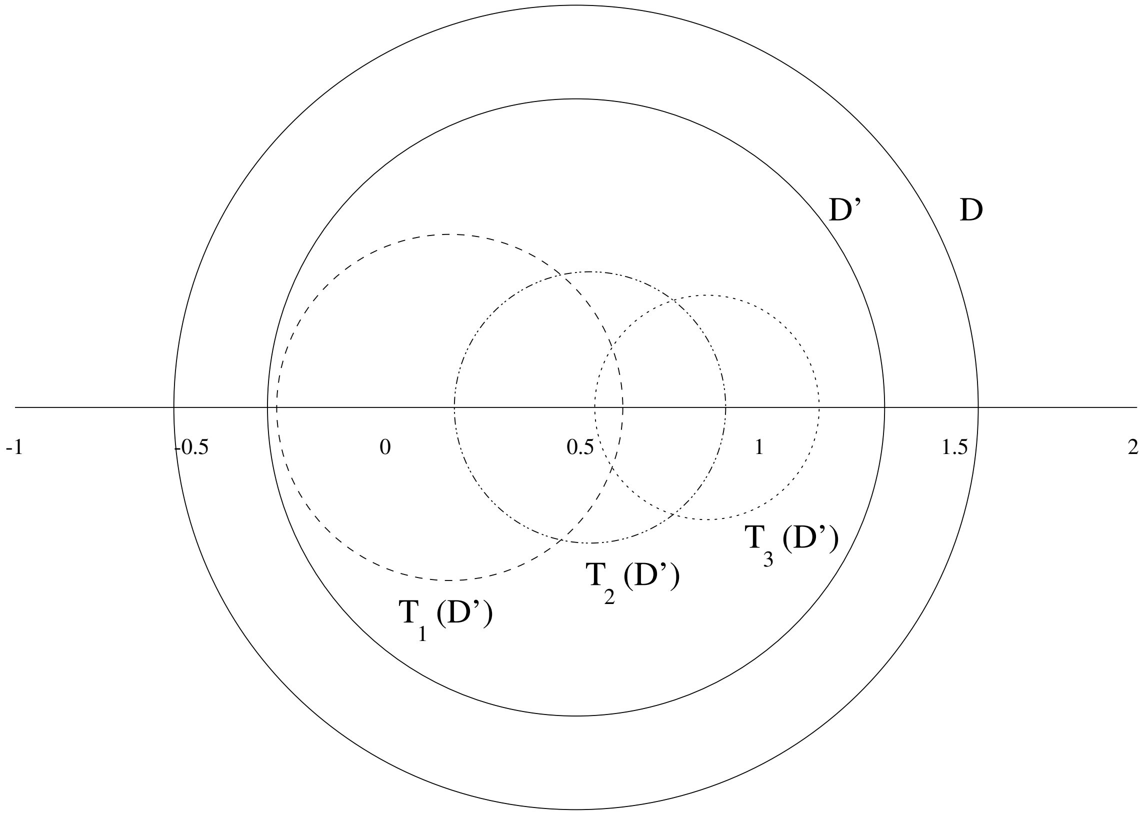

FIGURE. The disks D and D', and the images $T_i(D')$, $i = 1, 2, 3$.

The crucial consequence of this is that if $w \in A_\infty(D)$ then $\mathcal{L}_{g,t}w \in A_\infty(D')$. That is,

$$\mathcal{L}_{g,t}A_\infty(D) \subset A_\infty(D').$$

So the operator $\mathcal{L}_{g,t}$ *improves the regularity* of functions it operates on: if w is merely analytic on D, then its image $\mathcal{L}_{g,t}w$ is analytic on the larger disc D'.

We will exploit this in the proof of Proposition 3, where a circle of integration will be pushed arbitrarily close to the boundary of D'. A key quantity will be the ratio

$$\theta = \frac{1}{3-\sqrt{3}} \approx 0.788675\ldots$$

of the radii of the concentric discs $D \subset D'$. We call θ the *contraction ratio.*

For $\varepsilon > 0$ it will be convenient to denote

$$\theta_\varepsilon = \frac{\sqrt{3}/2}{\frac{3}{2}(\sqrt{3}-1)-\varepsilon},$$

and note that $\theta_\varepsilon \searrow \theta$ as $\varepsilon \to 0$.

Remark. *Our disc D was chosen so as to minimise the contraction ratio θ.*

Although the density function $\varrho := d\mu/d\lambda$ for the absolutely continuous invariant measure μ is not explicitly known, we have the following result.

Proposition 2. *The density function $\varrho = d\mu/d\lambda$ is real analytic on the interval $[0,1]$. Moreover, it has a holomorophic extension to the cut plane $\mathbb{C} \setminus (-\infty, -1]$.*

Proof. The fact that T is piecewise expanding, piecewise onto and piecewise analytic (all with respect to our Markov partition) means that ϱ is itself real analytic. This follows immediately from the well-known fact that ϱ is the fixed point of the Perron-Frobenius operator $\mathcal{L}$, and that this linear operator preserves the space of real analytic functions defined on I (see [PJ] for more details).

Let us suppose that the function ϱ is analytic on a domain $D \subset \mathbb{C}$ containing $[0,1]$. The three maps T_j are analytic on the cut plane $\mathbb{C} \setminus (-\infty, -1]$. Moreover, the same is true of the weight functions $1/(2\sqrt{z+j+1})$ in definition (3.1).

If $N_\varepsilon((-\infty,-1])$ denotes the ε-ball around the half-line $(-\infty,-1]$ then define $B(R,\varepsilon) = \{z \in \mathbb{C} : |z| < R\} \setminus N_\varepsilon((-\infty,-1])$. For n sufficiently large, each $T_j^n B(R,\varepsilon)$ lies inside the domain D, thus the compositions $\varrho \circ T_j^n$ are holomorphic on $B(R,\varepsilon)$, for $j = 0,1,2$. Therefore $\varrho = \mathcal{L}^n \varrho$ is also holomorphic on $B(R,\varepsilon)$. Letting $R \to \infty$ and $\varepsilon \to 0$ gives the result.

4. Fredholm determinants and nuclear operators

Letting $g^n(x) = g(x) + g(Tx) + \ldots g(T^{n-1}x)$, we define the following two-variable function $F = F_g$ in terms of our function g and the periodic points of T,

$$F(z,t) = F_g(z,t) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{T'_{\underline{i}}(x_{\underline{i}}) \exp\left(t g^n(x_{\underline{i}})\right)}{1 - T'_{\underline{i}}(x_{\underline{i}})} \right). \tag{4.1}$$

If D_1F, D_2F denote first and second partial derivatives, respectively, then we have

Lemma 2.

$$\int g\, d\mu = \frac{D_2F(1,0)}{D_1F(1,0)}$$

The proof of this Lemma is contained in [PJ]. The idea is to relate the integral to a certain derivative of topological pressure, which in turn is related to a zero of F.

Proposition 3. *For each $t \in \mathbb{C}$ there exist sequences of functions $v_n \in A_\infty(D)$ and functionals $l_n \in A_\infty(D)^*$ such that, for all $w \in \mathcal{A}_\infty(D)$,*

$$\mathcal{L}_{g,t} w = \sum_{n=1}^{\infty} l_n(w) v_n$$

where $||l_n||_\infty ||v_n||_\infty = O(\theta^n)$, and $\theta = \frac{1}{3-\sqrt{3}} \approx 0.788675\ldots$.

Proposition 3, together with the definition of a nuclear operator (see [Gr], [PJ]), immediately gives us:

Corollary 1. *For each $t \in \mathbb{C}$, the operator $\mathcal{L}_{g,t}$ is nuclear.*

Proof of Proposition 3. Let Γ_ε be the circle centred at $1/2$, of radius $\frac{3}{2}(\sqrt{3}-1)-\varepsilon$, where $\varepsilon > 0$. For ε sufficiently small we note that Γ_ε is outside of D but is just inside D'.

For $w \in A_\infty(D)$ we know that $L_{g,t}w \in A_\infty(D')$ (i.e. is analytic on the *larger* disc D'). So using Cauchy's theorem we have

$$\begin{aligned}\mathcal{L}_{g,t}w(z) &= \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(\mathcal{L}_{g,t}w)(\xi)}{\xi - z}d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(\mathcal{L}_{g,t}w)(\xi)}{\xi - 1/2}\left(1-\frac{z-1/2}{\xi-1/2}\right)^{-1}d\xi \\ &= \sum_{n=0}^{\infty} l_n(w)v_n(z),\end{aligned}$$

where

$$v_n(z) = (z-1/2)^n \qquad \text{and} \qquad l_n(w) = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(\mathcal{L}_{g,t}w)(\xi)}{(\xi-1/2)^{n+1}}d\xi.$$

Given that $z \in D$ and $\xi \in \Gamma$, we have

$$|(z-1/2)^n| \le \left(\frac{\sqrt{3}}{2}\right)^n \quad \text{and} \quad |\xi - 1/2|^{n+1} = \left(\frac{3}{2}(\sqrt{3}-1)-\varepsilon\right)^{n+1}.$$

Therefore, for all sufficiently small $\varepsilon > 0$, we have the estimate

$$\begin{aligned}||l_n||_\infty ||v_n||_\infty &\le \frac{||\mathcal{L}_{g,t}||_\infty}{2\pi(\frac{3}{2}(\sqrt{3}-1)-\varepsilon)}\left(\frac{\sqrt{3}/2}{\frac{3}{2}(\sqrt{3}-1)-\varepsilon}\right)^n \\ &= K_1||\mathcal{L}_{g,t}||_\infty \theta_\varepsilon^n,\end{aligned} \tag{4.2}$$

where $K_1 := 0.1449397969\ldots$.

Thus $||l_n||_\infty ||v_n||_\infty = O(\theta^n)$, as claimed.

Now we show that the integral $\int g\, d\mu$ can be effectively approximated.

Proposition 4. *For each $g \in A_\infty(D)$, there exists a real-valued sequence $S_n(g)$, whose n^{th} term is computable from the periodic points up to period n, such that*

$$\int g\, d\mu = S_n(g) + O(\theta^{n(n+1)/2}),$$

where $\theta = \frac{1}{3-\sqrt{3}} \approx 0.788675\ldots$.

Proof. Here we will only sketch the idea of the proof. Full details can be found in [PJ], and explicit estimates are carried out in sections 5 and 6, for particular choices of functions g.

The terms

$$\sum_{|\underline{i}|=n}\frac{T'_{\underline{i}}(x_{\underline{i}})\exp\left(tg^n(x_{\underline{i}})\right)}{1-T'_{\underline{i}}(x_{\underline{i}})}$$

in (4.1) are precisely the traces of the n^{th} iterate of our nuclear operator $\mathcal{L}_{g,t}$, which means that $F(z,t)$ is in fact the *Fredholm determinant* of $\mathcal{L}_{g,t}$ (see [Gr]), and in particular is an entire function of both $z \in \mathbb{C}$ and $t \in \mathbb{C}$.

Consequently there is an alternative form for $F(z,t)$ as the power series

$$F(z,t) = 1 + \sum_{n=1}^{\infty} C_n(t) z^n,$$

whose coefficients are given (see [Gr], p. 17) by

$$C_n(t) = \sum_{i_1 < \ldots < i_n} \lambda_{i_1} \ldots \lambda_{i_n} \operatorname{Det}\left[\left(m_{i_\alpha}(u_{i_\beta})\right)_{1 \le \alpha, \beta \le n}\right],$$

where $m_i := l_i/||l_i||_\infty$, $u_i := v_i/||v_i||_\infty$, and $\lambda_i := ||l_i||_\infty ||v_i||_\infty$ (all supremum norms are with respect to the disc D).

Since the entries in the matrix $\left(m_{i_\alpha}(u_{i_\beta})\right)_{1 \le \alpha, \beta \le n}$ are less than one in absolute value, its determinant is dominated in modulus by $n^{n/2}$, by a theorem of Hadamard [Had]. Our estimate (4.2) on the rate of decay of the λ_i implies

$$\begin{aligned} \sum_{i_1 < \ldots < i_n} \lambda_{i_1} \ldots \lambda_{i_n} &\le (K_1 ||\mathcal{L}_{g,t}||_\infty)^n \sum_{i_1 < \ldots < i_n} \theta^{i_1 + \ldots + i_n} \\ &= (K_1 ||\mathcal{L}_{g,t}||_\infty)^n \frac{\theta^{n(n+1)/2}}{(1-\theta)\ldots(1-\theta^n)}. \end{aligned}$$

Combining these gives the estimate

$$|C_n(t)| \le \frac{n^{n/2} (K_1 ||\mathcal{L}_{g,t}||_\infty)^n \theta^{n(n+1)/2}}{(1-\theta)\ldots(1-\theta^n)}. \tag{4.3}$$

Since $\prod_{n=1}^{\infty}(1-\theta^n) > 0$, then the expression in (4.3) is $O(\theta^{n(n+1)/2})$, (for more details see Lemma 8, [PJ]).

From this we deduce the same decay rate $O(\theta^{n(n+1)/2})$ for the power series coefficients of the partial derivatives $D_1 F$ and $D_2 F$ (see sections 5 and 6 for explicit estimates of these coefficients).

By Lemma 1 we can define $S_n(g)$ as the ratio of the degree-n polynomial truncations of $D_1 F$ and $D_2 F$ (evaluated at $z = 1$, $t = 0$).

Finally, we can use the series expansion of the exponential function in (4.1) to obtain an alternative form of the power series expansion for F, where this time the coefficients $C_n(t)$ are in terms of the periodic points of T. Namely we have

$$C_n(t) = \sum_{\substack{(j_1, \cdots, j_m) \\ j_1 + \ldots + j_m = n}} \frac{(-1)^m}{m!} a_{j_1}(t) \ldots a_{j_m}(t), \tag{4.4}$$

where

$$a_j(t) = \sum_{|\underline{i}| = j} \frac{T'_{\underline{i}}(x_{\underline{i}}) \exp\left(t g^n(x_{\underline{i}})\right)}{j(1 - T'_{\underline{i}}(x_{\underline{i}}))}. \tag{4.5}$$

Thus the $S_n(g)$ are also computable from the periodic points up to period n, as claimed.

5. Entropy

In this section we compute the entropy of the absolutely continuous invariant measure μ. To do this we fix our choice of analytic function g to be $g(x) = \log(2+2x)$, since this is the analytic extension of $\log|T'(x)|$. By the Rohlin-Pesin identity (see [Ro], [Pe]) we know that $h(\mu) = \int \log|T'|\,d\mu$.

Theorem 1. *The entropy $h(\mu)$ of the absolutely continuous invariant measure μ for the Bolyai-Rényi expansion is $h(\mu) = 1.05631307402 \pm 1.3 \times 10^{-7}$.*

Moreover, we have the empirical estimate $h(\mu) = 1.056313074 = \log 2.875748746$.

Proof.

Choosing $g(x) = \log|T'(x)| = \log(2+2x)$, we define $F(z,t)$ as in (4.1). We have series expansions

$$D_1F(1,0) = \sum_{n=1}^{\infty} nC_n(0) \quad , \quad D_2F(1,0) = \sum_{n=1}^{\infty} C_n'(0).$$

Using the formulae (4.4), (4.5) we can exactly compute the first 12 terms (our only limitation here is available computer power) in both of the above series. This gives us the order-12 approximations:

$$c_1 := \sum_{n=1}^{12} nC_n(0) = -0.393317995898306598279\ldots\,, \text{ and}$$

$$c_2 := \sum_{n=1}^{12} C_n'(0) = -0.4154669413176785127047\ldots\,.$$

By Lemma 2, this gives us the order-12 approximation

$$h(\mu) = \frac{D_2F(1,0)}{D_1F(1,0)} \approx \frac{c_2}{c_1} = 1.05631307402750669584133\ldots$$

To estimate the accuracy of this approximation we introduce the error terms $\varepsilon_i := |D_iF(1,0) - c_i|$, for $i = 1,2$.

First we estimate ε_1. To do this we will bound the norm $||\mathcal{L}_{g,t}||_\infty$ appearing in (4.2), for the case $t = 0$. That is, we will bound the norm $||\mathcal{L}||_\infty$ of the Perron-Frobenius operator. We have

$$||\mathcal{L}_{g,0}||_\infty = ||\mathcal{L}||_\infty \leq \sum_{j=0}^{2} \sup_{z\in D} \frac{1}{2\sqrt{z+j+1}}.$$

By the maximum modulus theorem, each of the above suprema is attained by some $z \in \partial D$, the circle of radius $\sqrt{3}/2$ around $1/2$. In fact it is easily seen that in each case the supremum is attained (uniquely) at the point $z = (1-\sqrt{3})/2$, so that

$$\begin{aligned} ||\mathcal{L}_{g,0}||_\infty = ||\mathcal{L}||_\infty &\leq \frac{1}{2\sqrt{\frac{1}{2}(3-\sqrt{3})}} + \frac{1}{2\sqrt{\frac{1}{2}(5-\sqrt{3})}} + \frac{1}{2\sqrt{\frac{1}{2}(7-\sqrt{3})}} \\ &\approx 0.6279630302 + 0.3911536204 + 0.308080502 \\ &= 1.327197153. \end{aligned}$$

Therefore we have

$$K_2 := K_1 ||\mathcal{L}||_\infty \leq 0.1449397969 \times 1.327197153 = 0.1923636858,$$

so that (4.3) gives

$$\begin{aligned}\varepsilon_1 =& \left| \sum_{n=13}^{\infty} nC_n(0) \right| \leq \sum_{n=13}^{\infty} \frac{n^{n/2} n K_2^n \theta^{n(n+1)/2}}{(1-\theta)\dots(1-\theta^n)} \\ =& 0.0000000080413605760125\dots \approx 8.04136 \times 10^{-9}\end{aligned}$$

Now we make the more delicate estimate of the error ε_2. This will involve estimating $||\mathcal{L}_{g,t}||_\infty$, for general $t \in \mathbb{C}$. Since our function is $g(x) = \log(2+2x)$, using (3.2) we have, for any $t \in \mathbb{C}$,

$$\begin{aligned}||\mathcal{L}_{g,t}||_\infty &\leq \sum_{j=0}^{2} \sup_{z \in D} \left| \frac{\left(1 + 2\sqrt{j+1+z}\right)^{-t}}{2\sqrt{j+1+z}} \right| \\ &= \sum_{j=0}^{2} \sup_{z \in \partial D} \left| \frac{\left(1 + 2\sqrt{j+1+z}\right)^{-t}}{2\sqrt{j+1+z}} \right| \\ &= \sum_{j=0}^{2} \max_{0 \leq \alpha \leq 1} \left| \frac{\left(1 + 2\sqrt{j + \frac{3}{2} + \frac{\sqrt{3}}{2} e^{2\pi i \alpha}}\right)^{-t}}{2\sqrt{j + \frac{3}{2} + \frac{\sqrt{3}}{2} e^{2\pi i \alpha}}} \right|,\end{aligned} \tag{5.1}$$

again by the maximum modulus principle.

Now we define

$$J(\delta) = K_1 \max_{|t|=\delta} \sum_{j=0}^{2} \max_{0 \leq \alpha \leq 1} \left| \frac{\left(1 + 2\sqrt{j + \frac{3}{2} + \frac{\sqrt{3}}{2} e^{2\pi i \alpha}}\right)^{-t}}{2\sqrt{j + \frac{3}{2} + \frac{\sqrt{3}}{2} e^{2\pi i \alpha}}} \right|. \tag{5.2}$$

By (5.1) we know $J(\delta) \geq K_1 ||\mathcal{L}_{g,t}||_\infty$ for all $t \in \mathbb{C}$ with $|t| = \delta$.

Since $F_g(z,t)$ is entire in t, by Cauchy's theorem we can estimate $C_n'(0)$ by integration around some suitable circle centred at 0. If this circle is of radius $\delta > 0$, then (4.3), (5.1), (5.2) give

$$\begin{aligned}|C_n'(0)| &= \left| \frac{1}{2\pi i} \int_{|t|=\delta} \frac{C_n(t)}{t^2} dt \right| \\ &\leq \frac{\max_{|t|=\delta} |C_n(t)|}{\delta^2 2\pi} \\ &\leq \frac{n^{n/2} J(\delta)^n \theta^{n(n+1)/2}}{\delta^2 2\pi (1-\theta)\dots(1-\theta^n)}.\end{aligned}$$

We then have

$$\varepsilon_2 = \left| \sum_{n=13}^{\infty} C_n'(0) \right| \leq \sum_{n=13}^{\infty} \frac{n^{n/2} J(\delta)^n \theta^{n(n+1)/2}}{\delta^2 2\pi (1-\theta)\dots(1-\theta^n)}, \tag{5.3}$$

for all $\delta > 0$, and it remains to make a suitable choice of δ to obtain the optimal upper bound on ε_2.

Choosing $\delta = 0.131$ (this gives close to the optimal estimate, given that we are using order-12 truncations) we can compute numerically that

$$\begin{aligned} J(0.131) &= 0.2240247382219096 = K_1 \times 1.545639934375747 \\ &= K_1\,(0.7114266780771788 + 0.4618830103302341 + 0.3723302459683343). \end{aligned}$$

(We remark that the suprema in the definition of $J(\cdot)$ are achieved for the value $t = -\delta = -0.131$, and the value $\alpha = 1/2$ (corresponding to the point $z = 1/2 - \sqrt{3}/2 \in \partial D$) for each $j = 0, 1, 2$).

In particular, for this choice of δ, (5.3) yields the estimate

$$\varepsilon_2 \leq 0.0000000417468846788179 8 \approx 4.1746 \times 10^{-8}.$$

We then have

$$\left| \frac{c_2}{c_1} - \frac{D_2F(1,0)}{D_1F(1,0)} \right| \leq \frac{|c_2|\varepsilon_1 + |c_1|\varepsilon_2}{|c_1|(|c_1| - \varepsilon_1)} \leq 0.00000012773654025328 39\ldots \approx 1.277 \times 10^{-7},$$

so our order-12 approximation is indeed within 1.3×10^{-7} of the true entropy.

To establish the empirical estimate, we compute the order-11 approximation (see Table 1) to the entropy as 1.056313074656433934604373.

Comparing this with the order-12 approximation, and observing (see Table 1) the speed of convergence, we obtain the empirical estimate $h(\mu) = 1.056313074$.

n	
5	1.058167707656605567182516 76
6	1.05615270306661367836155 44
7	1.05632514364487927086997 73
8	1.05631208106581883375996 28
9	1.05631315960102013568289 30
10	1.05631306692586559968826 6
11	1.05631307465643393460437 3
12	1.05631307402750669584133

TABLE 1. Successive approximations to the entropy $h(\mu)$ using periodic points up to period n

Remark. *In* [BDK] *it was conjectured (Conjecture 6.1) that* $h(\mu) \approx 1.0545$. *These authors use a statistical approach based on the use of the Shannon-McMillan-Breiman theorem, which is very different in nature from our approach.*

Corollary 2. *If* $x_N = f(a_1 + f(a_2 + \ldots + f(a_N)\ldots))$, *with* $f(x) = \sqrt{1+x} - 1$, *then for Lebesgue almost all points* x *we have* $|x - x_N| = O(0.34773557^N)$.

Proof. By Theorem 1, the true entropy $h(\mu)$ is at least 1.05631294, so by choosing some suitably small $\varepsilon > 0$ in part (2) of Proposition 1, we see that the exponential contraction rate of $|x - x_N|$ is $O(0.34773557^N)$, for almost every x.

6. Frequency of digits

By part (1) of Proposition 1, the frequencies with which the digits 0, 1, 2 appear in the Bolyai-Rényi expansion are well-defined (for Lebesgue almost every point), and given by integrating the characteristic functions χ_{I_i}, $i = 0, 1, 2$, with respect to the invariant measure μ. In this section we will accurately compute these frequencies.

Our method involves a slight modification of the one used to compute the entropy. Note that each χ_{I_i} does not belong to $\mathcal{A}_\infty(D)$. However, its restriction to each of the intervals I_j clearly is analytic, for $j = 0, 1, 2$. More formally, χ_{I_i} belongs to the space of functions which are analytic on the disjoint union $\coprod_{j=0}^2 I_j$. Indeed, taking the three separate analytic extensions of $\chi_{I_i}|_{I_j}$, $(j = 0, 1, 2)$, to D gives a function which is analytic on the disjoint union $\coprod_{j=0}^2 D_j$, where each D_j is a copy of D. This new function is simply χ_{D_i}. Moreover, the Fredholm determinant (defined by (4.1)) for $g = \chi_{I_i}$ coincides with that for $g = \chi_{D_i}$, since all period-n points $x_{\underline{i}}$ lie on I. Thus for calculational purposes we can work with the function χ_{D_i} rather than χ_{I_i}.

The important advantage of this is the following. Since each $\chi_{D_i} \circ T_j = 0$ or 1, then $\mathcal{L}_t^{(i)} := \mathcal{L}_{\chi_{D_i}, t}$ acts on $\mathcal{A}_\infty(D)$, and its image is contained in the space $\mathcal{A}_\infty(D')$ of functions analytic on the larger disc D' (by the same arguments as at the end of section 3). Consequently the analysis in section 4, and in particular the estimate (4.3), remains valid for $g = \chi_{D_i}$.

Explicitly, our transfer operators $\mathcal{L}_t^{(i)} := \mathcal{L}_{\chi_{D_i}, t}$ are defined by

$$\mathcal{L}_t^{(i)} w(z) = \frac{e^t w(T_i z)}{2\sqrt{z+i+1}} + \sum_{\substack{j=0 \\ j \neq i}}^{2} \frac{w(T_j z)}{2\sqrt{z+j+1}}. \tag{6.1}$$

Theorem 2. *For Lebesgue almost every $x \in I$, the frequency of occurrence of the digits 0, 1, 2 in the Bolyai-Rényi expansion of x are*

$$freq(0) = 0.46407962943 \pm 2.9 \times 10^{-8}$$
$$freq(1) = 0.3044190449 \pm 1.6 \times 10^{-8}$$
$$freq(2) = 0.2315013256 \pm 1.1 \times 10^{-8}$$

Moreover, we have the empirical estimates

$$freq(0) = 0.46407962943$$
$$freq(1) = 0.3044190449$$
$$freq(2) = 0.2315013256$$

Proof. By analogy with (4.1) we introduce the entire functions

$$\begin{aligned} F^{(j)}(z,t) &= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{T'_{\underline{i}}(x_{\underline{i}}) \exp\left(t\chi^n_{I_j}(x_{\underline{i}})\right)}{1 - T'_{\underline{i}}(x_{\underline{i}})} \right) \\ &= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{T'_{\underline{i}}(x_{\underline{i}}) \exp\left(t\chi^n_{D_j}(x_{\underline{i}})\right)}{1 - T'_{\underline{i}}(x_{\underline{i}})} \right) \qquad (6.2) \\ &= 1 + \sum_{n=1}^{\infty} C_n^{(j)}(t) z^n, \end{aligned}$$

for $j = 0, 1, 2$.

By Proposition 1 and Lemma 2 we can identify

$$freq(j) = \int \chi_{I_j} \, d\mu = \frac{D_2 F^{(j)}(1,0)}{D_1 F^{(j)}(1,0)},$$

where we have the series expansions

$$D_1 F^{(j)}(1,0) = \sum_{n=1}^{\infty} n C_n^{(j)}(0) \quad , \quad D_2 F^{(j)}(1,0) = \sum_{n=1}^{\infty} C_n^{(j)\prime}(0).$$

Note in fact that $D_1 F^{(j)}(1,0)$ is independent of j, so as in the proof of Theorem 1 we have the order-12 approximation:

$$c_1 := \sum_{n=1}^{12} n C_n^{(j)}(0) = -0.393317995898306598279,$$

for $j = 0, 1, 2$.

Defining the error term $\varepsilon_1 := |D_1 F^{(j)}(1,0) - c_1|$ (which is independent of j), by the same argument as in Theorem 1 we obtain the upper bound

$$\varepsilon_1 \leq 0.0000000080413605760125 \ldots \approx 8.04136 \times 10^{-9}$$

The relevant order-12 approximations for $D_2 F^{(j)}(1,0)$ are respectively

$$c_2^{(0)} := \sum_{n=1}^{12} C_n^{(0)\prime}(0) = -0.182530869785549059806378048463367126373 \ldots,$$

$$c_2^{(1)} := \sum_{n=1}^{12} C_n^{(1)\prime}(0) = -0.11973348866379947317802423837617903111 \ldots,$$

$$c_2^{(2)} := \sum_{n=1}^{12} C_n^{(2)\prime}(0) = -0.0910536374334888938007591366192137185 \ldots$$

By Lemma 2, this gives us the order-12 approximations

$$freq(0) = \frac{D_2 F^{(0)}(1,0)}{D_1 F^{(0)}(1,0)} \approx \frac{c_2^{(0)}}{c_1} = 0.4640796294323204473785420774671837071 \ldots$$

$$freq(1) = \frac{D_2 F^{(1)}(1,0)}{D_1 F^{(1)}(1,0)} \approx \frac{c_2^{(1)}}{c_1} = 0.3044190449265811915843610512988279987 \ldots$$

$$freq(2) = \frac{D_2 F^{(2)}(1,0)}{D_1 F^{(2)}(1,0)} \approx \frac{c_2^{(2)}}{c_1} = 0.2315013256017684253229890778051297912 \ldots$$

To estimate the accuracy of these approximation we introduce the error terms $\varepsilon_2^{(j)} := |D_2 F^{(j)}(1,0) - c_2^{(j)}|$, for $j = 0, 1, 2$.

To estimate the $\varepsilon_2^{(j)}$, we need to estimate $||\mathcal{L}_t^{(j)}||_\infty$. By analogy with the techniques used in the proof of Theorem 1, we define

$$\begin{aligned} J_{(0)}(\delta) &= K_1 \max_{|t|=\delta}\left(\max_{z\in\partial D}\left|\frac{e^t}{2\sqrt{z+1}}\right| + \max_{z\in\partial D}\left|\frac{1}{2\sqrt{z+2}}\right| + \max_{z\in\partial D}\left|\frac{1}{2\sqrt{z+3}}\right|\right) \\ &= K_1\left(\frac{e^\delta}{2\sqrt{\frac12(3-\sqrt3)}} + \frac{1}{2\sqrt{\frac12(5-\sqrt3)}} + \frac{1}{2\sqrt{\frac12(7-\sqrt3)}}\right) \end{aligned} \tag{6.3}$$

Note that $J_{(0)}(\delta) \ge K_1||\mathcal{L}_t^{(0)}||_\infty$ for all $t\in\mathbb{C}$ with $|t|=\delta$.

Defining $J_{(1)}(\delta), J_{(2)}(\delta)$ analogously, we derive the explicit expressions

$$J_{(1)}(\delta) = K_1\left(\frac{1}{2\sqrt{\frac12(3-\sqrt3)}} + \frac{e^\delta}{2\sqrt{\frac12(5-\sqrt3)}} + \frac{1}{2\sqrt{\frac12(7-\sqrt3)}}\right),$$

$$J_{(2)}(\delta) = K_1\left(\frac{1}{2\sqrt{\frac12(3-\sqrt3)}} + \frac{1}{2\sqrt{\frac12(5-\sqrt3)}} + \frac{e^\delta}{2\sqrt{\frac12(7-\sqrt3)}}\right).$$

As in the proof of Theorem 1, we can estimate $C_n^{(j)\prime}(0)$ by integrating around a circle of radius δ, centred at 0. By (4.3), (6.3) we have

$$\begin{aligned} |C_n^{(j)\prime}(0)| &= \left|\frac{1}{2\pi i}\int_{|t|=\delta}\frac{C_n^{(j)}(t)}{t^2}dt\right| \\ &\le \frac{\max_{|t|=\delta}|C_n^{(j)}(t)|}{\delta^2 2\pi} \\ &\le \frac{n^{n/2}J_{(j)}(\delta)^n\theta^{n(n+1)/2}}{\delta^2 2\pi(1-\theta)\dots(1-\theta^n)}. \end{aligned}$$

We then have

$$\varepsilon_2^{(j)} = \left|\sum_{n=13}^\infty C_n^{(j)\prime}(0)\right| \le \sum_{n=13}^\infty \frac{n^{n/2}J_{(j)}(\delta)^n\theta^{n(n+1)/2}}{\delta^2 2\pi(1-\theta)\dots(1-\theta^n)}, \tag{6.4}$$

for all $\delta>0$, and it remains to make a suitable choice of δ to obtain the optimal upper bound on $\varepsilon_2^{(j)}$. The choise of δ will be different for each $j=0,1,2$.

For $j=0$ we choose $\delta=0.280538$, which gives

$$J_{(0)}(0.280538) = 0.22183874550802618353,$$

and (6.4) gives the upper bound

$$\varepsilon_2^{(0)} \le 0.0000000080094010700419\ldots \approx 8.0094\times10^{-9}.$$

This gives us

$$\left|\frac{c_2^{(0)}}{c_1} - \frac{D_2F(1,0)}{D_1F(1,0)}\right| \le \frac{|c_2^{(0)}|\varepsilon_1 + |c_1|\varepsilon_2^{(0)}}{|c_1|(|c_1|-\varepsilon_1)} \le 0.000000029851756260\ldots \approx 2.9851\times10^{-8},$$

so our order-12 approximation is indeed within 2.9×10^{-8} of the true frequency $freq(0)$.

For $j = 1$ we choose $\delta = 0.4$, which gives $J_{(1)}(0.4) = 0.22024706058611192697$, and (6.4) gives the upper bound

$$\varepsilon_2^{(1)} \leq 0.0000000035863066455352 31\ldots \approx 3.5863 \times 10^{-9}.$$

We then have

$$\left|\frac{c_2^{(1)}}{c_1} - \frac{D_2F(1,0)}{D_1F(1,0)}\right| \leq \frac{|c_2^{(1)}|\varepsilon_1 + |c_1|\varepsilon_2^{(1)}}{|c_1|(|c_1| - \varepsilon_1)} \leq 0.0000000153419119854\ldots \approx 1.5341 \times 10^{-8},$$

so our order-12 approximation is indeed within 1.6×10^{-8} of the true frequency $freq(1)$.

For $j = 2$ we choose $\delta = 0.47$, which gives $J_{(2)}(0.47) = 0.21915530173992111521$, and (6.4) gives the upper bound

$$\varepsilon_2^{(2)} \leq 0.00000000243447427742 1805\ldots \approx 2.4344 \times 10^{-9}.$$

We then have

$$\left|\frac{c_2^{(2)}}{c_1} - \frac{D_2F(1,0)}{D_1F(1,0)}\right| \leq \frac{|c_2^{(2)}|\varepsilon_1 + |c_1|\varepsilon_2^{(2)}}{|c_1|(|c_1| - \varepsilon_1)} \leq 0.0000000109226123468\ldots \approx 1.0922 \times 10^{-8},$$

so our order-12 approximation is indeed within 1.1×10^{-8} of the true frequency $freq(2)$.

The empirical estimate for each $freq(j)$ is given by comparing the 11^{th} and 12^{th} order approximations in Tables 2, 3 and 4, and retaining those digits which coincide.

n	
5	0.46435909501454506686307566929087458078
6	0.46406049499842454303522745347088830886
7	0.46408186730089402715550697909797236095
8	0.46407938130821526467334701452249103314
9	0.46407965347225846507480290244178162010
10	0.4640796272949911418861341265524892711
11	0.4640796296310131100783392003010399183
12	0.4640796294323204473785420774671837071

Table 2. Successive approximations to $freq(0)$ using periodic points up to period n

Corollary 3. *For almost every $x \in I$, with digit sequence $a_1, a_2, \ldots$, the asymptotic arithmetic mean is*

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + \ldots + a_n) = 0.767421696 \pm 0.56 \times 10^{-8}$$

n	
5	0.30503305810398604203711128821431036127
6	0.30436504181880606086969695969237938890
7	0.30442298977744286102431668914647584683
8	0.30441873023402450744848124413377540095
9	0.30441907163931442404009449037405814340
10	0.3044190427311780262387140577206270104
11	0.3044190451195828345302730034394649730
12	0.3044190449265811915843610512988279987

Table 3. Successive approximations to $freq(1)$ using periodic points up to period n

n	
5	0.23197531592304682578341152207518262438
6	0.23145978794196133643466232097577697555
7	0.23150435225320446593755544220404095829
8	0.23150108449559544202968446721419088614
9	0.23150134604955240614907489089593967961
10	0.2315013239226018884286778845661499860
11	0.2315013257492966889337542141808042650
12	0.2315013256017684253229890778051297912

Table 4. Successive approximations to $freq(2)$ using periodic points up to period n

Proof. Taking $f = \sum_{j=0}^{2} j\chi_{I_j}$, Birkhoff's ergodic theorem gives the empirical estimate

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + \ldots + a_n) = \int f\, d\mu = freq(1) + 2freq(2) = 0.767421696,$$

by the empirical results of Theorem 2. The rigorous error bound is given by summing the three error bounds in Theorem 2.

7. Rate of Correlation Decay

In this final section we compute empirically the rate of correlation decay of analytic functions (observables) for the Bolyai-Rényi dynamical system (T, μ).

For analytic functions f, g, define the n^{th} *correlation function*

$$c_n(f,g) = \int f \cdot T^n g\, d\mu - \int f\, d\mu \int g\, d\mu.$$

Lemma 3. *The exponential rate of correlation decay*

$$\sup\left\{ \limsup_{n\to\infty} \sqrt[n]{c_n(f,g)} : f, g \in \mathcal{A} \right\}$$

is given by the second-largest (in modulus) eigenvalue α of the Perron-Frobenius operator $\mathcal{L}$.

Proof. First note that

$$\begin{aligned}\int f \cdot T^n g \, d\mu &= \int f(T^n x) g(x) \varrho(x) \, dx \\ &= \int f(x) \mathcal{L}^n (g\varrho)(x) \, dx,\end{aligned}$$

by the change of variable formula, where $\varrho = \mathcal{L}\varrho$ denotes the density function $d\mu/d\lambda$.

Writing $g = c + (g - c)$, where $c = \int g \, d\mu$, we have

$$\begin{aligned}\mathcal{L}^n(g\varrho) &= \mathcal{L}^n(c\varrho) + \mathcal{L}^n\left((g-c)\varrho\right) \\ &= c\varrho + \mathcal{L}^n\left((g-c)\varrho\right) \\ &= c\varrho + \varrho \int (g-c) \, d\mu + O(\alpha^n) \\ &= \varrho \int g \, d\mu + O(\alpha^n),\end{aligned}$$

with respect to the uniform norm (see Ruelle [Ru2]). Therefore

$$\begin{aligned}\int f(x) \mathcal{L}^n (g\varrho)(x) \, dx &= \int f(x) \left(\varrho(x) \int g \, d\mu + O(\alpha^n) \right) \, dx \\ &= \int f \, d\mu \int g \, d\mu + O(\alpha^n),\end{aligned}$$

as required.

The fact that $\mathcal{L}$ is nuclear (set $t = 0$ in Proposition 3) means its eigenvalues (with algebraic multiplicities) are precisely the reciprocals of the zeros (again with algebraic multiplicities) of the Fredholm determinant $d(z) = F(z, 0) = 1 + \sum_{m=1}^{\infty} C_m(0) z^m$, see [Gr]. In particular, the modulus α_n of the second smallest zero of the polynomial truncation $d_n(z) = 1 + \sum_{m=1}^{n} C_m(0) z^m$ converges to the rate of correlation decay α.

The α_n are listed in Table 5, for $n = 1, \ldots, 12$.

Theorem 3. *The rate of correlation decay of the Bolyai-Rényi expansion, with respect to the absolutely continuous invariant measure μ, is* 0.40582 ± 0.000076.

Moreover, this rate of correlation decay is given empirically as 0.40582.

Proof. We will use Rouché's theorem to compare the zeros of the Fredholm determinant $d(z)$ to those of its twelth order approximation $d_{12}(z)$. The first twelve coefficients $C_1(0), \ldots, C_{12}(0)$ are given in Table 6.

More precisely, for suitable choices of circular contour Γ_i we will show that

$$|d(z) - d_{12}(z)| < |d_{12}(z)| \quad \text{for all } z \in \Gamma_i, \tag{7.1}$$

and deduce that d, d_{12} have the same number of zeros in the disc enclosed by Γ_i.

The term $|d(z) - d_{12}(z)| = \sum_{m=13}^{\infty} C_m(0)z^m$ is estimated by means of the bound (4.3).

The three leading zeros (i.e. zeros of minimum modulus) of d_{12} are

$$
\begin{aligned}
z_1 &= 0.99999999999559060219576663208556880538719, \\
z_2 &= 2.46409306770808208005653362549270757380082, \\
z_3 &= 5.20333607178584420947630332332064588062949.
\end{aligned}
$$

Taking Γ_1 to be the circle centred at $\frac{1}{2}(z_1 + z_2)$, and of radius $3/4$, say, we easily check that (7.1) holds and deduce that the Fredholm determinant d has exactly two zeros in the disc A_1 bounded by Γ_1.

Now one of these zeros is clearly the value 1 (the reciprocal of the dominant eigenvalue for the Perron-Frobenius operator), so the other zero of d in the disc A_1, let us call it ζ_2, will determine the rate of correlation decay $\alpha = 1/|\zeta_2|$.

Taking Γ_2 to be the circle centred at z_2, of radius 0.00046, the bound (4.3) gives that $\max_{z\in\Gamma_2} |d(z) - d_{12}(z)| < 0.0000815564$. But $\min_{z\in\Gamma_2} |d_{12}(z)| > 0.0000824847$, so that (7.1) again holds. Therefore d has exactly one zero in the disc A_2 bounded by Γ_2. Since $A_2 \subset A_1$ then this zero must be ζ_2. Thus we have $|\zeta_2 - z_2| < 0.00046$.

Therefore

$$
\left| \frac{1}{\zeta_2} - \frac{1}{z_2} \right| = \frac{|\zeta_2 - z_2|}{z_2(z_2 - 0.00046)} \le 0.0000757748,
$$

which implies that $|\alpha - \alpha_{12}| < 0.000076$, as required.

For the empirical estimate on the rate of correlation decay we compare the values of α_{11} and α_{12} in Table 5.

n	
5	0.435653145324223844999088153863946064596
6	0.397102616453987736051654117525214584356
7	0.406920873040137604802158035015422868446
8	0.405649389292433446633945589173233989276
9	0.405861265806557501370475653273750134816
10	0.40582321246632835070475457314019671126
11	0.40583004615124056172705350044844807446
12	0.40582882728943609329427667842912235016

TABLE 5. Successive approximations to the rate of correlation decay

REFERENCES

[Bl] F. Blanchard, *β-expansions and symbolic dynamics*, Theor. Comp. Sci. **65** (1989), 131–141.

[Bol] W. Bolyai, *Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris nethodo intuitiva evidentiaque huic propria introducendi, ed. sec., Vol. 1*, Budapest, 1897.

[BDK] W. Bosma, K. Dajani, and C. Kraaikamp, *Entropy and counting correct digits*, preprint (1999).

m	$C_m(0)$
1	-1.7805469288332912726151680670795885804214570052556384781875
2	0.968150202409978866190931425171238046258873082241796047534
3	-0.200449499881595422668099261082517627176267155823210543450
4	0.011747548689539614515696836102251495253538538089366139039
5	0.001320729616885514423440075737825684210946681816268338684
6	-0.00023845939808201314790525094806777333962592582638710399
7	0.000017377487824315663577513619371577982425956328823478337
8	-0.000001034099171687213435387495713380529939649979420805256
9	0.000000068690346747371206658806483281475344799644156631040
10	-0.00000000502354667531727959402600313654340414663735337001
11	0.00000000036586403391418295704578835805521809268377948334
12	-0.0000000000264863166246360636702410679182409538505237410

TABLE 6. First 12 power series coefficients $C_m(0)$ of the Fredholm determinant $d(z) = F(z, 0)$

[Fried] D. Fried, *The zeta functions of Ruelle and Selberg I*, Ann. Sci. ENS **19** (1986), 491–517.

[Gr] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[Had] J. Hadamard, *Résolution d'une question relative aux déterminants*, Bull. Sci. Math. **28** (1893), 240–246.

[LaY] A. Lasota and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[Pa] W. Parry, *On the β-expansions of real numbers*, Acta. Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[Pe] J. B. Pesin, *Lyapunov characteristic exponents and smooth ergodic theory*, Russ. Math. Surveys **32** (1977), 55–114.

[PJ] M. Pollicott and O. Jenkinson, *Computing invariant densities and metric entropy*, preprint (1999).

[Ren] A. Rényi, *Representations of real numbers and their ergodic properties*, Acta. Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[Ro] V. A. Rohlin, *Exact endomorphism of a Lebesgue space*, Amer. Math. Soc. Transl. **39** (1964), 1–36.

[Ru1] D. Ruelle, *Zeta functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), 231–242.

[Ru2] D. Ruelle, *Thermodynamic Formalism*, Wiley, New York, 1978.

[Sch] F. Schweiger, *Ergodic theory of fibred systems and metric number theory*, The Clarendon Press, Oxford University Press, New York, 1995.

[Wal] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

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