COHOMOLOGY CLASSES OF DYNAMICALLY NON-NEGATIVE C^k FUNCTIONS

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Abstract. Let $T : x \mapsto 2x \pmod 1$ be the doubling map of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We construct a trigonometric polynomial $f : \mathbb{T} \to \mathbb{R}$ with the following property: $\int f\,d\mu \geqslant 0$ for every T-invariant probability measure μ, so that f is cohomologous to a non-negative Lipschitz function, yet f is not cohomologous to any non-negative C^1 function.

1. Introduction

Given a continuous transformation $T : X \to X$ of a compact metric space X, let $\mathcal{M}$ denote the set of T-invariant Borel probability measures on X, and $C(X)$ the space of real-valued continuous functions on X, equipped with the uniform topology. A function on X is called a (continuous) *coboundary* if it can be written as $\varphi - \varphi \circ T$ with $\varphi \in C(X)$. Two functions are said to be (continuously) *cohomologous* if their difference is a coboundary; this is clearly an equivalence relation on $C(X)$, and we are interested in its equivalence classes, the (continuous) *cohomology classes*.

Clearly, if f is a coboundary then we have $\int f\mu = 0$ for any $\mu \in \mathcal{M}$. A celebrated theorem of Livšic [L1,L2] states that the converse is also true, provided that the dynamical system (X, T) is hyperbolic and f is sufficiently regular; moreover, in this case the cobounding function φ has the same regularity as $f = \varphi - \varphi \circ T$.

Theorem (Livšic). *Let $T : \mathbb{T} \to \mathbb{T}$ be a C^ω expanding map. Let $f : \mathbb{T} \to \mathbb{R}$ be C^k for some $k = 1, 2, \ldots, \infty, \omega$ (resp. β-Hölder for some $0 < \beta \leqslant 1$), and suppose $\int f\mu = 0$ for all $\mu \in \mathcal{M}$. Then there exists a C^k (resp. β-Hölder) function $\varphi \in C(\mathbb{T})$ such that $f = \varphi - \varphi \circ T$.*

Livšic' theorem in fact applies to more general uniformly hyperbolic dynamical systems, but the case of expanding maps of the circle is sufficient for our needs. In fact Livšic' theorem is usually stated with the a priori weaker hypothesis that $\int f\mu = 0$ whenever μ is a periodic orbit, but this actually makes no difference, by a classical result of Parthasarathy [Pa] and Sigmund [Sig] stating that periodic orbits are dense in $\mathcal{M}$ for the weak* topology.

Actually Livšic only proved the "Hölder" part of the above theorem. He gives some partial results on the "differentiable" part, but leaves the general C^k problem as an open question. Livšic also gives an analogue of the above theorem for Anosov

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flows. These results were rediscovered independently by Guillemin & Kazhdan [GK1,GK2] in their study of isospectral rigidity.

Livšic' theorem relies very much on the hyperbolicity of the dynamical system. In the non-hyperbolic setting, there is no such general result about the cohomological equation $\varphi - \varphi \circ T = f$. In the special case where T is an irrational rotation of the circle, the cohomological equation is essentially a problem of small denominators (writing f and φ as Fourier series), and the existence and smoothness of the solutions depend on the arithmetic properties of the rotation number. Livšic' theorem also requires a regularity hypothesis on f (the Hölder condition is sufficient, though it can be weakened) but for reasons which are harder to explain; we will see in §3 that arbitrary continuous f have a very different, and in a sense pathological, behaviour with respect to cohomology.

Because of Livšic' theorem, we can define a C^k coboundary in two different but equivalent ways: either as a C^k function which is a coboundary, or as a function of the form $\varphi - \varphi \circ T$ where φ is a C^k function. We can similarly define Hölder coboundaries, and from this we derive the notions of C^k and Hölder cohomology classes in the obvious way.

Henceforth we shall restrict ourselves to the case where T is a C^ω expanding (i.e. $|T'| > 1$ everywhere) map of the circle $\mathbb{T}$. In particular T is topologically transitive, so the equation $\varphi - \varphi \circ T = f$ can have at most one solution, up to an additive constant.

In this article we consider a problem analogous to that of Livšic. Suppose that $f, \varphi \in C(\mathbb{T})$ are such that $f \geqslant \varphi - \varphi \circ T$; then clearly we have $\int f\mu \geqslant 0$ for all $\mu \in \mathcal{M}$. Here again we can ask whether the converse, or a partial converse, is true. One can prove the following:

Theorem A. *Let $T : \mathbb{T} \to \mathbb{T}$ be a C^1 expanding map. Let $f \in C(\mathbb{T})$ be a β-Hölder function for some $0 < \beta \leqslant 1$, and suppose $\int f\mu \geqslant 0$ for all $\mu \in \mathcal{M}$. Then there exists a β-Hölder function $\varphi \in C(\mathbb{T})$ such that $f \geqslant \varphi - \varphi \circ T$.*

This theorem has been stated and proved independently by several people. It first appears in an unpublished manuscript by Conze & Guivarc'h [CG], where it is proved using thermodynamic formalism; the same approach is used by Savchenko [S]. More direct proofs, which do not use the Ruelle transfer operator, can be found in [B1,B2,CLT].

Another way to state Theorem A is by introducing the minimum ergodic average $\alpha(f)$, defined by $\alpha(f) = \min_{\mu \in \mathcal{M}} \int f\mu$. The theorem then says that if $\alpha(f) \geqslant 0$ (we shall say that f is *dynamically non-negative* in this case), T is expanding and f is Hölder, then f is cohomologous to a non-negative Hölder function.

Theorem A is closely related to the structure of the set of *minimizing* measures, i.e., those $\mu \in \mathcal{M}$ such that $\int f\mu = \alpha(f)$. It implies in particular (as the reader will readily verify) the so-called *subordination principle* [B2]: under the hypotheses of Theorem A, if $\mu, \nu \in \mathcal{M}$ are such that $\operatorname{supp} \mu \subset \operatorname{supp} \nu$, and ν is minimizing, then μ is also minimizing. The Hölder part of Livšic' theorem can also be easily derived from Theorem A [B2].

However, the differentiable case is conspicuously absent from Theorem A, and one might wonder whether it is merely an artefact of the methods used to prove it. A priori, it would be reasonable to conjecture, by analogy with Livšic' theorem, that if T, f are C^k (for some $k = 1, 2, \ldots, \infty, \omega$) and $\alpha(f) \geqslant 0$, then $f \geqslant \varphi - \varphi \circ T$ for some C^k function φ. Indeed this result was claimed in an unpublished preprint

[PS][1], written at a time when very little was known about minimizing measures, or about the possible obstructions to the above statement (and variants of it). Soon afterwards, one of us [J1, J2] conjectured a model describing the minimizing measures for the family of functions $f_\theta : x \mapsto \cos 2\pi(x-\theta)$, parametrized by $\theta \in \mathbb{R}/\mathbb{Z}$, and where T is the angle-doubling map. In this model, for some values of θ there is a minimizing measure whose support is a Cantor subset of $\mathbb{T}$. For such values of θ there cannot exist a real-analytic φ satisfying $f - \alpha(f) \geqslant \varphi - \varphi \circ T$, for $f - \alpha(f)$ and $\varphi - \varphi \circ T$ would then coincide on the support of this measure, a non-discrete subset of $\mathbb{T}$, and thus be equal, easily leading to a contradiction. This model (later proved in [B1]), shows that there is no C^ω analogue of Theorem A and, as we shall see in this article, can also provide an obstruction to φ being C^k for arbitrary k, indeed even for $k = 1$.

In the following theorem, T is again the angle-doubling map, and f is a degree one trigonometric polynomial such that $\alpha(f) = 0$, so that T and f are both real-analytic. By Theorem A, with $\beta = 1$, we know that f is cohomologous to a non-negative Lipschitz function. On the other hand, however, Theorem B says that f is *not* cohomologous to a non-negative C^1 function. This counterexample shows that Theorem A cannot be extended to the differentiable setting.

Theorem B. *Let $T : x \mapsto 2x \pmod 1$ be the angle-doubling map on the circle $\mathbb{T}$. The trigonometric polynomial $f : \mathbb{T} \to \mathbb{R}$ defined by*

$$\begin{aligned} f(x) &= 1 - \cos 2\pi x - C \sin 2\pi x \\ &= 1 - \sqrt{1 + C^2} \cos 2\pi (x - \omega) \end{aligned}$$

where

$$\begin{aligned} C &= \frac{\sum_{n=0}^\infty 1 - \cos(\pi/2^n)}{\sum_{n=0}^\infty \sin(\pi/2^n)} = 1.368231157\ldots \\ \omega &= \frac{\arctan C}{2\pi} = 0.149550073\ldots \end{aligned}$$

satisfies $\alpha(f) = \min_{m\in\mathcal{M}} \int f\, dm = 0$, yet it is not cohomologous to a C^1 non-negative function.

2. Proof of Theorem B

For any real parameter C, define the function $f_C : \mathbb{T} \to \mathbb{R}$ by

$$\begin{aligned} f_C(x) &= 1 - \cos 2\pi x - C \sin 2\pi x \\ &= 1 - \sqrt{1 + C^2} \cos 2\pi (x - \omega) \end{aligned}$$

where $\omega = (2\pi)^{-1} \arctan C$. Clearly $f_C(0) = 0$ for all C. We claim that there exists a unique $C \in \mathbb{R}$ such that f is cohomologous to a function which vanishes on the semicircle $[0, 1/2]$. (In this case we say that f is cohomologous to zero on $[0, 1/2]$).

[1]This preprint was not submitted for publication, as Pollicott & Sharp discovered it was incorrect soon after it was circulated.

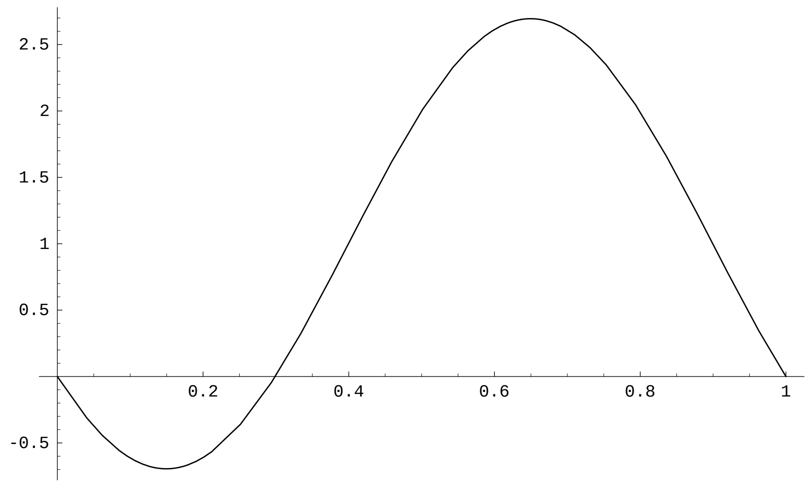

FIGURE 1. Graph of the function f in Theorem B

Lemma 1. *A Lipschitz function $f : \mathbb{T} \to \mathbb{R}$ is cohomologous to zero on the closed semicircle $[0, 1/2]$ if and only if $f(0) = 0$ and $\sum_{n=1}^{\infty} f(2^{-n}) = 0$.*

Proof. Suppose there exists $\Phi \in C(\mathbb{T})$ such that the function $g = \Phi - \Phi \circ T + f$ is zero on the semicircle $[0, 1/2]$. From $g(0) = 0$ we get $f(0) = 0$, and the relation $0 = \sum_{n=1}^{N} g(2^{-n})$ can be rewritten as

$$0 = \Phi(2^{-N}) - \Phi(0) + \sum_{n=1}^{N} f(2^{-n})$$

Letting $N \to \infty$, we obtain $0 = \sum_{n=1}^{\infty} f(2^{-n})$; so the conditions are necessary.

Conversely, suppose $f(0) = 0$ and $\sum_{n=1}^{\infty} f(2^{-n}) = 0$. The Lipschitz condition ensures that the series $\varphi(x) = \sum_{n=1}^{\infty} f(x2^{-n})$ is uniformly convergent on every compact interval, so that φ is a continuous function on $\mathbb{R}$. The condition $\sum_{n=1}^{\infty} f(2^{-n}) = 0$ gives $\varphi(1) = 0 = \varphi(0)$, so we can consider the restriction of φ to $[0, 1]$ as a continuous function on the circle $\mathbb{T}$, which we denote by Φ. We then have, for all $x \in [0, 1/2]$, $\Phi(Tx) - \Phi(x) = \varphi(2x) - \varphi(x) = f(x)$. So the function $f + \Phi - \Phi \circ T$ is identically zero on $[0, 1/2]$; the condition is sufficient, and the lemma is proved.

Remarks. We see that if f is C^k, or real-analytic, or an entire function, then so is φ. However, Φ will not be C^1 in general; $\varphi'(0)$ and $\varphi'(1)$ will usually differ, so the left and right derivatives of Φ will disagree at the origin.

Lemma 2. *Let $f : \mathbb{T} \to \mathbb{R}$ be as in the statement of Theorem B. Then f is cohomologous to a non-negative function g satisfying*

(1) *g is identically zero on $[0, 1/2]$,*
(2) *g is strictly positive on $(1/2, 1)$,*
(3) *g is differentiable at all points except 0 and $1/2$,*
(4) *The righthand derivative $R = g'_+(1/2)$ at the point $1/2$ is strictly positive.*

Proof. The constant C in the statement of Theorem B was chosen precisely to ensure that $\sum_{n=1}^{\infty} f(2^{-n}) = 0$, so Lemma 1 applies. If we define φ, Φ and g as in

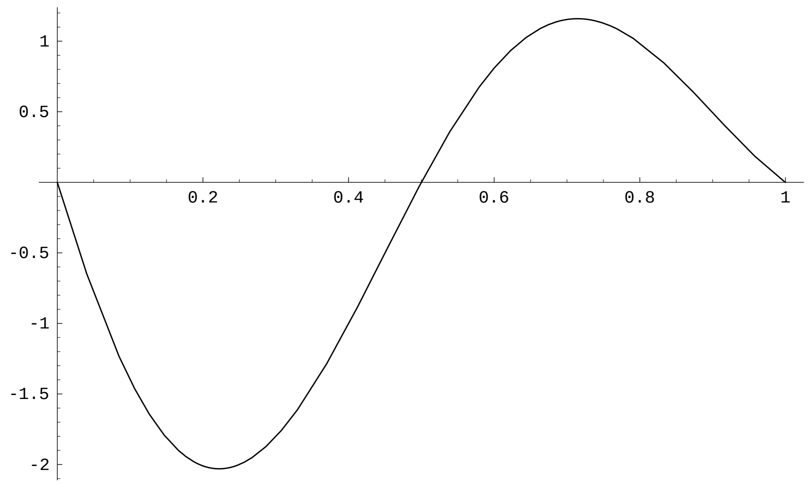

FIGURE 2. Graph of φ, restricted to $[0,1]$

the proof of Lemma 1 (φ is shown in Figure 2), then we know that g is identically zero on $[0,1/2]$.

When $x \in [1/2,1]$, we have

$$\begin{aligned} g(x) &= \Phi(x) - \Phi(2x) + f(x) = \varphi(x) - \varphi(2x-1) + f(x) \\ &= \varphi(2x) - \varphi(2x-1), \end{aligned}$$

so the restriction of g to $[1/2,1]$ is smooth (see the graph in Figure 3), with derivative $g'(x) = 2[\varphi'(2x) - \varphi'(2x-1)]$. At the endpoints, we have

$$\begin{aligned} g'_+(\tfrac{1}{2}) &= 2[\varphi'(1) - \varphi'(0)] = 26.8703\ldots > 0 \\ g'_-(1) &= 2[\varphi'(2) - \varphi'(1)] = -13.4351\ldots < 0. \end{aligned}$$

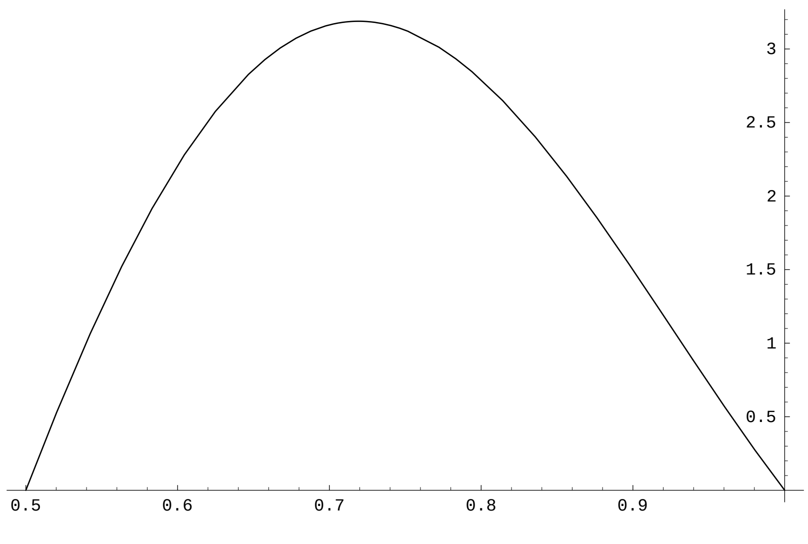

FIGURE 3. Graph of g restricted to $[1/2,1]$

The proof that $g(x) > 0$ for all $x \in (1/2,1)$ will use the following interpolation lemma, which is a special case of a classical result on Hermite interpolation, see e.g. [SB], Theorem (2.1.5.10), p. 56. We provide a proof for completeness.

Lemma 2a. *Let $a < b$, and suppose $u : [a,b] \to \mathbb{R}$ is a C^4 function such that $u(a) = u(b) = 0$. For all $x \in (a,b)$, there exists $\zeta \in (a,b)$ such that*

$$\frac{u(x)}{(x-a)(b-x)} = \frac{(b-x)u'(a) - (x-a)u'(b)}{(b-a)^2} + (x-a)(b-x)\frac{u^{(4)}(\zeta)}{24}$$

Proof of lemma 2a. Let $x \in (a,b)$, and define

$$P(t) = (t-a)(b-t)\left[\frac{(b-t)u'(a) - (t-a)u'(b)}{(b-a)^2} + (t-a)(b-t)\frac{\theta}{24}\right]$$

where the constant θ is chosen such that $P(x) = u(x)$. The function $v = u - P$ satisfies $v(a) = v'(a) = v(x) = v(b) = v'(b) = 0$, so it has (at least) five zeros (counted with multiplicity) and a repeated application of Rolle's theorem shows that there exists $\zeta \in (a,b)$ such that $v^{(4)}(\zeta) = 0$. Since $P^{(4)} = \theta$ identically, we obtain $u^{(4)}(\zeta) = \theta$, and the lemma is proved.

To apply Lemma 2a, we need estimates on the fourth derivative of g. First we note that

$$\begin{aligned}\varphi^{(4)}(x) &= \sum_{n=1}^{\infty} 16^{-n} f_C^{(4)}(x/2^n) \\ &= (2\pi)^4\sqrt{1+C^2}\sum_{n=1}^{\infty} 16^{-n}\cos 2\pi(x/2^n - \omega)\end{aligned}$$

and consequently

$$\left|\varphi^{(4)}(x)\right| \leqslant \frac{1}{15}(2\pi)^4\sqrt{1+C^2}$$

for all $x \in \mathbb{R}$. Now $g^{(4)}(x) = 16[\varphi^{(4)}(2x) - \varphi^{(4)}(2x-1)]$ whenever $1/2 < x < 1$, thus

$$\left|g^{(4)}(x)\right| \leqslant \frac{32}{15}(2\pi)^4\sqrt{1+C^2} = 5634.75\ldots$$

Now applying Lemma 2a to the function g, with $[a,b] = [1/2, 1]$, we see that, for all $x \in (1/2, 1)$,

$$\begin{aligned}\frac{g(x)}{(x-\frac12)(1-x)} &= 4\big\{(1-x)g'_+(\tfrac12) - (x-\tfrac12)g'_-(1)\big\} + (x-\tfrac12)(1-x)\frac{g^{(4)}(\zeta)}{24} \\ &\geqslant 4\big\{26(1-x) + 13(x-\tfrac12)\big\} - 235(x-\tfrac12)(1-x) \\ &\geqslant 21\end{aligned}$$

which completes the proof of Lemma 2.

An immediate consequence of Lemma 2 is that f has a unique minimizing measure, the Dirac mass at the origin, δ_0. In particular, $\alpha(f) = 0$.

Let $\theta_{r,n} = 2^r/(2^n - 1)$ (with $0 \leqslant r \leqslant n-1$) denote the points on the periodic orbit generated by $1/(2^n - 1)$, and let $R = g'_+(1/2) > 0$. We observe that these orbits are "almost" minimizing, in the following sense: the sum

$$\sum_{r=0}^{n-1} f(\theta_{r,n}) = \sum_{r=0}^{n-1} g(\theta_{r,n}) = g(\theta_{n-1,n}) \sim R\,(\theta_{n-1,n} - \tfrac12) \sim R/2^{n+1}$$

tends to zero exponentially fast as $n \to \infty$.

Now we are ready to prove Theorem B. Suppose that f is cohomologous to a non-negative $h \in C^1(\mathbb{T})$. Since h is cohomologous to g, which vanishes on $[0, 1/2]$, we see by Lemma 1 that $h(0) = 0$ and $\sum_{n=1}^{\infty} h(2^{-n}) = 0$. But the function h is non-negative, so it must vanish on the set $Z = \{0\} \cup \{2^{-n} : n \geqslant 1\}$. Since h is C^1, all its zeros must be "double" zeros, that is, the derivative h' must also vanish on Z (see Figure 4). Since h' is continuous, this implies that

$$h(x) = o\big(d(x, Z)\big)$$

in the sense that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, Z) \leqslant \delta$ implies $h(x) \leqslant \varepsilon d(x, Z)$, where d denotes the usual distance function on $\mathbb{T}$.

We have $d(\theta_{r,n}, Z) \leqslant \theta_{r,n} - 2^{r-n} = 2^{r-n}/(2^n - 1) \leqslant 1/2(2^n - 1)$, so $d(\theta_{r,n}, Z)$ tends to 0 uniformly in r as $n \to \infty$, and consequently

$$\sum_{r=0}^{n-1} f(\theta_{r,n}) = \sum_{r=0}^{n-1} h(\theta_{r,n}) = o\left(\sum_{r=0}^{n-1} d(\theta_{r,n}, Z)\right) = o(2^{-n})$$

as $n \to \infty$, which contradicts our previous estimate of $\sum_{r=0}^{n-1} f(\theta_{r,n})$, and concludes the proof of Theorem B.

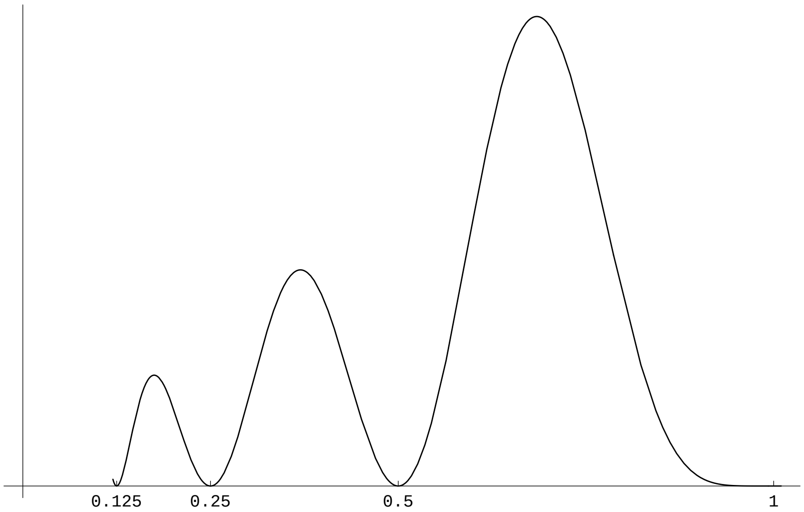

FIGURE 4. Schematic graph of h

3. WEAK COHOMOLOGY CLASSES FOR CONTINUOUS FUNCTIONS

Both Livšic' theorem and Theorem A require f to be Hölder. While this regularity hypothesis can be weakened [B2], it cannot be dispensed with completely. In this section, we will study what happens when f is an arbitrary continuous function, and T is the angle-doubling map. We shall see that both theorems fail, as does a weaker form of Theorem A.

Lemma 3. *A function $f \in C(\mathbb{T})$ satisfies $\int f \mu = 0$ for all $\mu \in \mathcal{M}$ if and only if f is a uniform limit of coboundaries.*

Proof. It is easily verified that the condition $\int f \mu = 0$ for all $\mu \in \mathcal{M}$ is equivalent to $S_n f/n \to 0$ in $C(\mathbb{T})$, where $S_n f = f + f \circ T + \cdots + f \circ T^{n-1}$ (see e.g. [CG], théorème

2.1). Moreover, f is cohomologous to $S_n f/n$, so $\int f\mu = 0\ \forall \mu \in \mathcal{M}$ implies that f is the uniform limit of the sequence $f_n = f - S_n f/n$ of coboundaries. The converse is obvious.

A function which satisfies either of the conditions above will be called a *weak coboundary.* Coboundaries are not closed in $C(\mathbb{T})$, so weak coboundaries form a larger (and closed) vector subspace of $C(\mathbb{T})$. There are several ways to see this, but the most convincing one is to construct an explicit weak coboundary which is not a coboundary; for the following construction we are indebted to M. Zinsmeister (private communication).

Exceptionally we will consider complex-valued functions on $\mathbb{T}$, since the Fourier series notation is more concise. Let $(a_n)_{n\geqslant 0}$ be a sequence of complex numbers such that $\sum |a_n| < \infty$ and $\sum a_n = 0$, and such that $\sum |r_n|^2 = \infty$, where $r_n = \sum_{p\geqslant n} a_p$. For example we could take $r_n = 1/\sqrt{n}$ for $n > 0$ and $r_0 = 0$, and then set $a_n = r_n - r_{n+1}$. Finally, define $E(t) = \exp(2\pi i t)$.

We then claim that $f(t) = \sum_{n=0}^{\infty} a_n E(2^n t)$ is a weak coboundary, but not a coboundary. It is a weak coboundary because it is the uniform limit as $m \to \infty$ of $\varphi_m(t) - \varphi_m(2t)$, where $\varphi_m(t) = \sum_{n=0}^{m-1} r_{n+1} E(2^n t)$. To prove that f is not a coboundary, assume $f(t) = \varphi(t) - \varphi(2t)$ for some $\varphi \in C(\mathbb{T})$. Considering φ as an element of $L^2(\mathbb{T})$, we can expand it as a Fourier series, and by identification of the coefficients in $f(t)$ and $\varphi(t) - \varphi(2t)$ we obtain $\varphi(t) = \sum_{n=0}^{\infty} r_{n+1} E(2^n t)$. We then see that the Fourier coefficients of φ are not ℓ^2, which is absurd. So f is not a coboundary.

For such a function f, which is a weak coboundary but not a coboundary, we clearly have $\alpha(f) = 0$. However there can be no $\varphi \in C(\mathbb{T})$ such that $f \geqslant \varphi - \varphi \circ T$, for it would imply the equality of f and $\varphi - \varphi \circ T$ (indeed, both functions have the same integral with respect to Lebesgue measure), contradicting our hypothesis that f is not a coboundary. This means that both Livšic' theorem and Theorem A are not valid for arbitrary continuous functions.

However, one might ask if perhaps a weaker form of Theorem A is true. Say that two functions f, g are *weakly cohomologous* if their difference is a weak coboundary, that is, if $\int f\mu = \int g\mu$ for all $\mu \in \mathcal{M}$.

Question. *Let T be the angle-doubling map, and let $f \in C(\mathbb{T})$ be such that $\alpha(f) \geqslant 0$. Is f weakly cohomologous to a non-negative function?*

An affirmative answer to this question was claimed in [Sh1,Sh2], but both proofs are incorrect. In fact the above question has a *negative* answer.

To understand this, we have to make explicit the link between the above assertion and the subordination principle. Suppose that g is a non-negative function with $\alpha(g) = 0$; its minimizing measures are those $\mu \in \mathcal{M}$ such that $\operatorname{supp}\mu \subset g^{-1}(0)$. Therefore the subordination principle applies to g: if μ, ν are two invariant measures such that $\operatorname{supp}\mu \subset \operatorname{supp}\nu$ and ν is minimizing, then μ is also minimizing. Now if f is weakly cohomologous to g, then it has the same minimizing measures, so the subordination principle also holds for f. In particular, f cannot have a fully supported minimizing measure, unless f is a weak coboundary (this point had already been made by Sharp).

On the other hand, it turns out that a *generic* (in the uniform topology) $f \in C(\mathbb{T})$ *does* have a fully supported minimizing measure! This obviously provides an obstruction to $f - \alpha(f)$ being weakly cohomologous to a non-negative function, by the above arguments. This assertion, that generically the minimizing measure has full support in $\mathbb{T}$, is claimed without proof in [B2], so we provide a proof here.

Lemma 4. *Let K be a proper closed subset of $\mathbb{T}$, and let $C_{\min}(K)$ be the set of functions $f \in C(\mathbb{T})$ which have a minimizing measure μ with $\operatorname{supp}\mu \subset K$. Then $C_{\min}(K)$ is a closed subset of $C(\mathbb{T})$ with empty interior.*

Proof. Replacing K by its maximal invariant subset $\bigcap_{n\geqslant 0} T^{-n}K$, we may assume, without loss of generality, that K is invariant; that is, $TK \subset K$.

It is easily verified that $C_{\min}(K)$ is closed. Now assume that $C_{\min}(K)$ has an interior point f_0; up to translation, we can assume $\alpha(f_0) = 0$. Denote by $\mathcal{M}(K)$ the set of $\mu \in \mathcal{M}$ such that $\operatorname{supp}\mu \subset K$. Let $\mu_0 \in \mathcal{M}(K)$ be ergodic and such that $\int f_0\mu_0 = 0$. Let $\varepsilon > 0$ be arbitrary.

It is easy to see that μ_0 can be approximated, in the weak* topology on $\mathcal{M}$, by periodic orbits which are not in $\mathcal{M}(K)$, so let $\mu \in \mathcal{M} - \mathcal{M}(K)$ be a periodic orbit such that $\int f_0\mu \leqslant \varepsilon$. Since $\operatorname{supp}\mu$ and K are disjoint, we can find a Urysohn function $g \in C(\mathbb{T})$ such that $g(x) = 0$ for all $x \in K$, $g(x) = 1$ for all $x \in \operatorname{supp}\mu$, and $0 \leqslant g(x) \leqslant 1$ everywhere. Therefore the function $f = f_0 - 2\varepsilon g$ satisfies $\|f - f_0\| = 2\varepsilon$ and $\int f\mu \leqslant -\varepsilon$. On the other hand, for any $\nu \in \mathcal{M}(K)$ we have $\int f\nu = \int f_0\nu \geqslant \alpha(f_0) = 0$, so such a ν cannot be a minimizing measure for f. Thus $f \notin C_{\min}(K)$. Moreover, f is arbitrarily close to f_0, contradicting our hypothesis that f_0 is an interior point of $C_{\min}(K)$. The lemma is proved.

Theorem C. *For a generic $f \in C(\mathbb{T})$, every minimizing measure of f has full support.*

Proof. Let $U_0, U_1 \ldots$ be a countable base of open subsets of $\mathbb{T}$, for instance, the open intervals with rational endpoints. For any $n \in \mathbb{N}$, the set Ω_n of functions $f \in C(\mathbb{T})$ which do not have a minimizing measure whose support avoids U_n is, by Lemma 4, an open and dense subset of $C(\mathbb{T})$. Their intersection $\bigcap \Omega_n$ consists of those f all of whose minimizing measures have full support, and is a dense G_δ subset of $C(\mathbb{T})$ by Baire's theorem.

Actually, it is also known that a generic $f \in C(\mathbb{T})$ has a unique minimizing measure [B2,CG] but we did not want to use this argument, which does not simplify the proof.

Summarising all the above arguments, we conclude:

Corollary. *For a generic $f \in C(\mathbb{T})$, there is no non-negative function which is weakly cohomologous to $f - \alpha(f)$.*

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