The flat spot standard family: variation of the entrance time median

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Abstract. We completely determine the median of the entrance time function for the one-parameter standard family of flat spot maps. The function mapping parameters to medians is shown to be a countably piecewise affine degree-one homeomorphism, with uncountably many cusps and precisely nine critical points.

1. Introduction

A *flat spot map* $T : \mathbb{T} \to \mathbb{T}$ (see e.g. [**2, 5, 11, 16, 17, 18, 19**]) is defined to be a continuous degree-one map of the circle $\mathbb{T}$ with the property that $T(F) = v$ for some point $v \in \mathbb{T}$ and open interval F (the 'flat spot'), with $T|_{\mathbb{T}\setminus\overline{F}}$ an expanding diffeomorphism onto its image $\mathbb{T} \setminus \{v\}$. For such maps, *first entrance time functions* $e_T(x) := \inf\{i \geq 0 : T^i(x) \in F\}$ were studied in [**2**] (though see e.g. [**3, 7, 8, 9, 12, 14**] for first entrance time functions in other contexts), with particular focus on the *standard family* $(T_\gamma)_{\gamma\in\mathbb{T}}$ (see also e.g. [**6, 16, 17, 18**]), defined (cf. Figure 1) by

$$T_\gamma(x) = \begin{cases} 2\gamma \pmod 1 & \text{for } x \in F_\gamma := (\gamma + 1/2, \gamma)\,, \\ 2x \pmod 1 & \text{for } x \in [\gamma, \gamma + 1/2]\,. \end{cases}$$

Of interest is the way in which these entrance time functions vary with the parameter γ. The graphs in Figure 2 suggest that, in some sense, the mass of e_{T_γ} moves in the direction of increasing γ. A means of making precise this intuition is via the *entrance time median* $\omega(\gamma)$, defined implicitly by[1] $\int_\gamma^{\omega(\gamma)} e_{T_\gamma} = \int_{\omega(\gamma)}^{\gamma+1/2} e_{T_\gamma}$, the unique point in $(\gamma, \gamma + 1/2)$ which divides equally the mass of e_{T_γ}. The purpose of this paper is to perform a complete analysis of the entrance time median function $\omega : \mathbb{T} \to \mathbb{T}$, confirming that indeed $\omega(\gamma)$ does move in the direction of increasing γ, but that it does so in a rather complicated and interesting way (cf. Figure 3, although much of the interesting detail described below is not visible in that graph):

[1]Throughout we shall use $\int g$ to denote the Lebesgue integral of a function g on the whole circle $\mathbb{T}$, and $\int_A g$ or $\int_a^b g$ for its Lebesgue integral over the set A or interval $[a, b]$.

THEOREM 1.1. *The entrance time median map $\omega : \mathbb{T} \to \mathbb{T}$ is a degree-1 homeomorphism which is countably piecewise affine (i.e. there is a countable partition of $\mathbb{T}$ into intervals, and ω is affine on each of these). It has precisely 9 critical points (points of zero derivative), and uncountably many cusps (points of infinite derivative).*

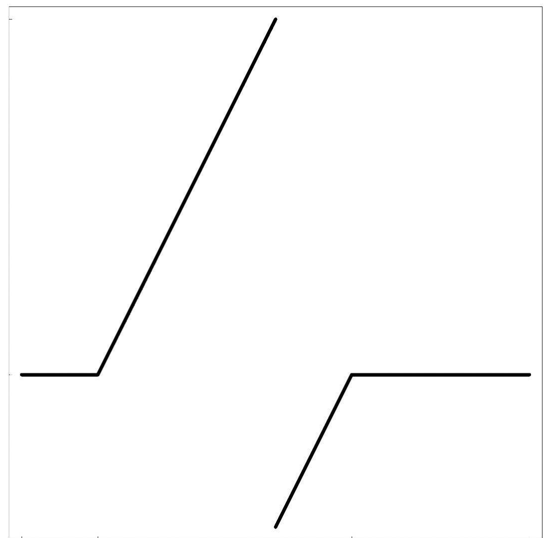

FIGURE 1. A standard flat spot map

REMARK 1.2.
(i) The map $\omega : \mathbb{T} \to \mathbb{T}$ is graphed in Figure 3; note the rotational symmetry about the point $(1/4, 1/2)$. The nine critical points of ω can be naturally identified with nine exceptional rotation numbers $0, \pm 2/7, \pm 3/10, \pm 1/3, \pm 3/8$ for the family $(T_\gamma)_{\gamma \in \mathbb{T}}$ (see Proposition 3.5, Remark 3.6, Theorem 3.13). The critical point at $-1/4 \equiv 3/4$ is clearly visible in Figure 3, while the other eight critical points are not (though see the proof of Proposition 3.5, and Remark 3.6, for their exact locations).
(ii) The uncountably many cusps of ω occur at those γ for which the rotation number $\varrho(T_\gamma)$ is irrational (see Theorem 3.13).
(iii) The intervals of parameter space on which ω is affine fall naturally into two categories. The first category consists of intervals $\Gamma(p/q) := \{\gamma \in \mathbb{T} : \varrho(T_\gamma) = p/q\}$ for $p/q \neq 0, \pm 2/7, \pm 3/10, \pm 1/3, \pm 3/8$ (each such interval has strictly positive length, cf. Lemma 2.1). The second category consists of countably many intervals which themselves form a partition of $\Gamma(p/q)$ in the exceptional cases $p/q = 0, \pm 2/7, \pm 3/10, \pm 1/3, \pm 3/8$; these intervals naturally accumulate, from either side, on the corresponding critical point mentioned in (i) above.
(iv) We are able to provide *explicit formulae* for all image intervals $\Omega(p/q) := \omega(\Gamma(p/q))$ (see Section 4); in particular, the endpoints of these intervals are always rational.

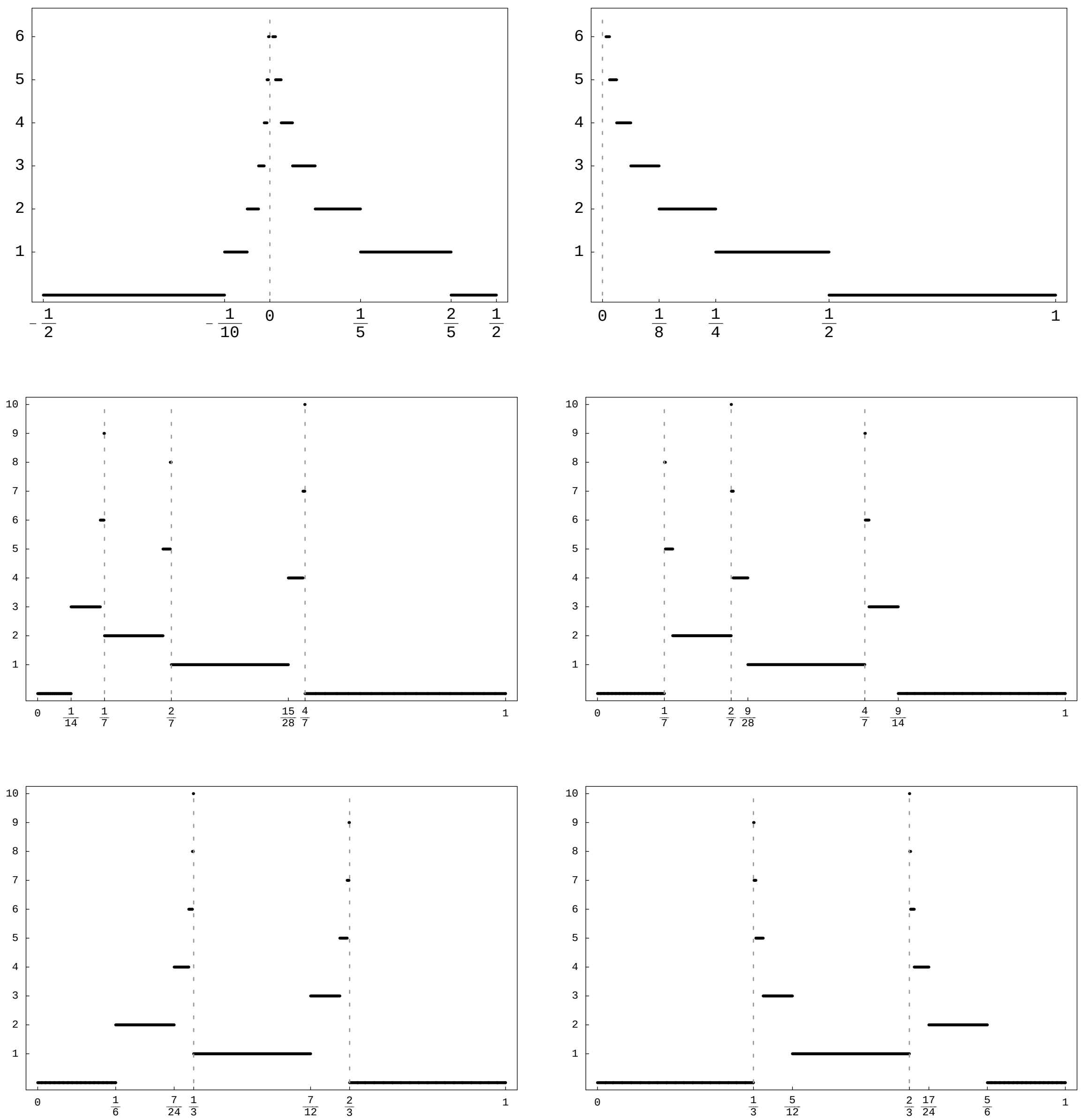

FIGURE 2. Evolution of entrance time functions e_{T_γ}, corresponding (from top left) to parameter values $\gamma = -1/10$, $\gamma = 0$, $\gamma = 1/14$, $\gamma = 1/7$, $\gamma = 1/6$, $\gamma = 1/3$.

For example $\Omega(0) = [-1/8, 1/8]$, $\Omega(1/2) = [7/18, 11/18]$, $\Omega(1/3) = \left[\frac{527}{1960}, \frac{251}{784}\right]$, and (cf. Corollary 4.4)

$$\Omega(1/q) = \left[\frac{2^{2q-1} + (2+3q)2^{q-2} - 1}{4(2^q-1)^2}, \frac{2^{2q-1} + (1+3q)2^{q-1} - 1}{4(2^q-1)^2}\right] \quad \text{for } q \geq 4\,. \tag{1}$$

The explicit formula for the general interval $\Omega(p/q)$ is more complicated (see Theorems 4.3 and 4.7), and involves the permutation induced by rotation by angle p/q; the relative simplicity of (1) reflects the relative simplicity of this permutation when $p = 1$. For example the formulae for the intervals $\Omega(1/2)$ and $\Omega(1/3)$ are *not* recovered by simply setting $q = 2, 3$ in (1).

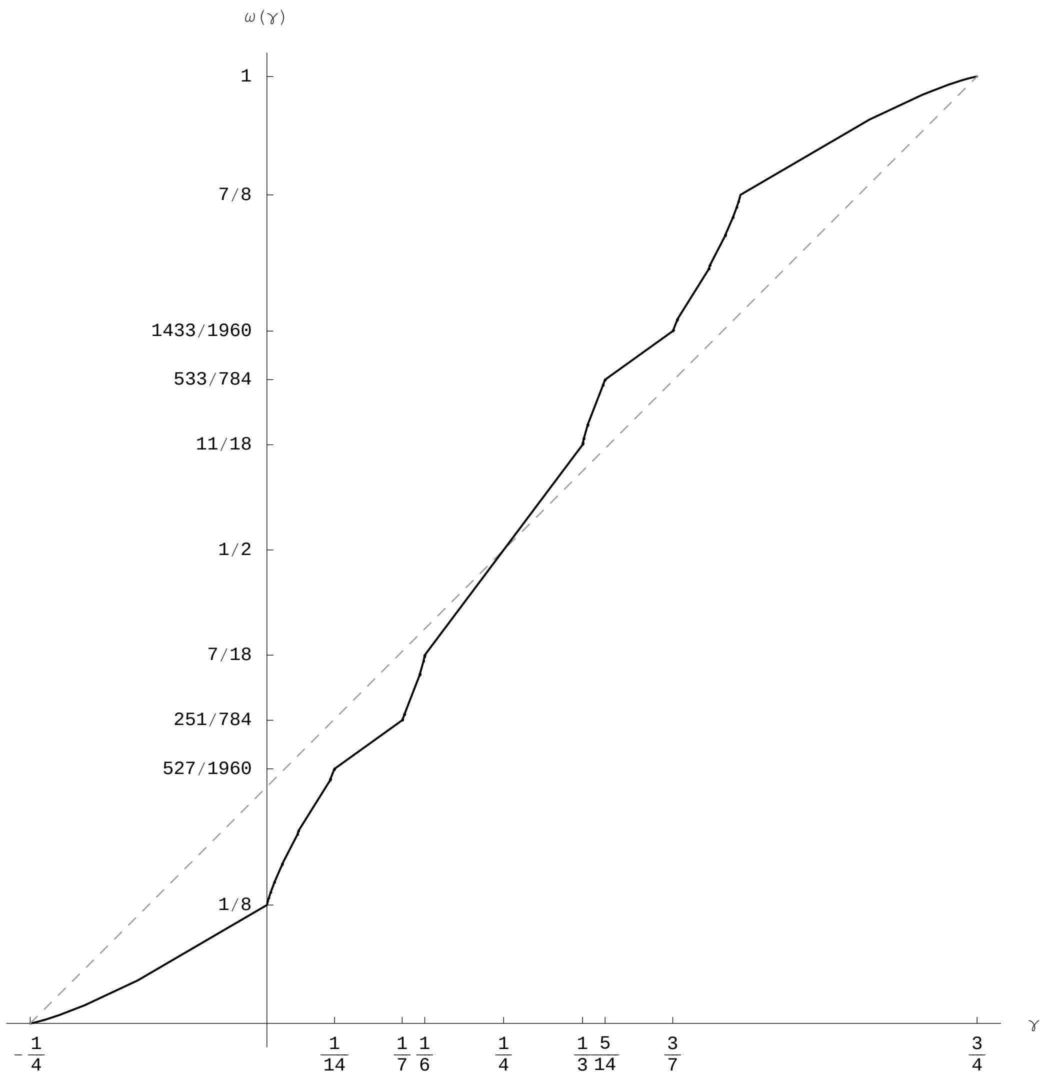

FIGURE 3. Graph of the entrance time median homeomorphism $\omega : \mathbb{T} \to \mathbb{T}$

2. Preliminaries

For a closed T-invariant subset $A \subset \mathbb{T}$, we say $x \in \mathbb{T}$ is *absorbed* by A if there exists $n = n(x) \in \mathbb{N}$ such that $T^n(x) \in A$. Such an A is called *totally absorbing* if every $x \in \mathbb{T}$

is absorbed by A, and *absorbing* if all but finitely many $x \in \mathbb{T}$ are absorbed by A. The map T is said to be *totally absorbing* if some proper T-invariant subset of $\mathbb{T}$ is totally absorbing. The following facts about the flat spot standard family are contained in [**2**]:

LEMMA 2.1. *If $\varrho(T_\gamma)$ is irrational then there exists a totally absorbing T_γ-invariant Cantor subset of $\mathbb{T} \setminus F_\gamma$. If $\varrho(T_\gamma)$ is rational then either a single periodic orbit $\mathcal{O}_{p/q}$ is totally absorbing, or there are precisely two periodic orbits, one of which is absorbing and absorbs all points of $\mathbb{T}$ except those points in the other periodic orbit $\mathcal{O}_{p/q}$.*

The map $\gamma \mapsto \varrho(\gamma) := \varrho(T_\gamma)$ is a continuous, weakly increasing, degree-1 self-map of $\mathbb{T}$, and its graph is a devil's staircase: the preimage of any rational rotation number p/q is a positive-length closed interval $\varrho^{-1}(p/q) =: [\gamma_{\min}(p/q), \gamma_{\max}(p/q)]$, while the preimage of any irrational is a singleton. Indeed $\{\gamma \in \mathbb{T} : T_\gamma$ is totally absorbing$\}$ is a Cantor subset of $\mathbb{T}$ with zero Hausdorff dimension. In particular, $\{\gamma \in \mathbb{T} : \varrho(\gamma) \notin \mathbb{Q}\}$ has zero Hausdorff dimension, and hence zero Lebesgue measure.

The T_γ-invariant Cantor sets of Lemma 2.1, and the periodic orbits $\mathcal{O}_{p/q}$ which are common to all T_γ for $\gamma \in \Gamma(p/q) = \varrho^{-1}(p/q)$, are known as *Sturmian*[2] sets; the dynamics of T_γ restricted to these invariant sets is combinatorially equivalent to the relevant rigid rotation.

We shall write the q points in $\mathcal{O}_{p/q}$ as $s_1 < s_2 < \ldots < s_q$, with $<$ the induced ordering on $(0,1)$. In other words $s_i = s_i(p/q)$ denotes the i-th largest point in $\mathcal{O}_{p/q}$ with respect to the ordering induced by $(0,1)$.

EXAMPLE 2.2. If $\gamma \in \Gamma(2/5) = [9/62, 5/31]$ then the points in the T_γ-periodic orbit $\mathcal{O}_{2/5}$ are $s_1 = 5/31$, $s_2 = 9/31$, $s_3 = 10/31$, $s_4 = 18/31$, and $s_5 = 20/31$.

DEFINITION 2.3. Define the *first entrance time function* $e_\gamma = e_{T_\gamma} : \mathbb{T} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $e_\gamma(x) = \inf\{i \geq 0 : T_\gamma^i(x) \in F_\gamma\}$, and define $E_n(\gamma) := e_\gamma^{-1}(n)$ for $n \geq 0$.

DEFINITION 2.4. For the Sturmian periodic orbit $s_1 < \ldots < s_q$ of rotation number p/q, define its *orbit partition* to consist of the q intervals $[s_q, s_1]$, $[s_1, s_2], \ldots, [s_{q-1}, s_q]$. In other words (cf. [**2**]), this partition consists of closed intervals $K_0, K_1, \ldots, K_{q-1}$, where each K_l has length $2^{q-1-l}(2^q - 1)^{-1}$, with the property that T maps K_l affinely onto K_{l-1} for $1 \leq l \leq q-1$. If $J_i := [s_i, s_{i+1}]$ for $1 \leq i \leq q-1$ then $\{J_i\}_{i=1}^{q-1}$ and $\{K_l\}_{l=1}^{q-1}$ are different indexings of the same collection of intervals, namely the orbit partition with the largest piece removed, and if $\pi_{p,q} : \{1, 2, \ldots, q-1\} \to \{1, 2, \ldots, q-1\}$ is defined by $\pi_{p,q}(l) = -lp \pmod q$, then $K_l = J_{\pi_{p,q}(l)}$ for $1 \leq l \leq q-1$.

NOTATION 2.5. The *length* of an interval I is its Lebesgue measure, denoted by $|I|$, and we define $|I|_\gamma := \int_I e_\gamma$. Given non-empty sub-intervals I and I' of an interval

[2]The terminology *Sturmian* is due to Morse & Hedlund [**13**], who studied the 0-1 sequences corresponding to the binary expansions of points in Sturmian sets. These *Sturmian sequences* have since then been extensively investigated, see e.g. [**1, 4, 6, 10, 15, 17**] for further characterisations and properties.

$J \subset \mathbb{T}$, we say that I is *to the left of* I', and that I' is *to the right of* I, and write $I \prec I'$, if $x \le x'$ for all $x \in I$, $x' \in I'$, where $\le$ denotes the ordering on J.

The following key technical result from [**2**, Thm. 3.13, Cor. 4.4, Cor. 4.6] gives an explicit description of the entrance time function e_γ on each orbit partition piece K_l, $0 \le l \le q-1$.

LEMMA 2.6. *Suppose that T_γ has rational rotation number p/q, with (Sturmian) periodic orbit $s_1 < \ldots < s_q$ outside F_γ, and $0 \le l \le q-1$. Then the interval K_l can be written as $K_l = \overline{\bigcup_{i\in\mathbb{Z}} E_l^i(\gamma)}$, where the $E_l^i(\gamma)$ are pairwise disjoint intervals whose relative ordering in K_l is $E_l^i(\gamma) \prec E_l^{i+1}(\gamma)$ for all $i \in \mathbb{Z}$, and $E_{l+iq}(\gamma) = E_l^{-i}(\gamma) \cup E_l^i(\gamma)$ for $i \ge 0$. In particular, $E_l(\gamma) = E_l^0(\gamma)$ is an interval, and $e_\gamma \equiv l + |i|q$ on $E_l^i(\gamma)$ for $i \in \mathbb{Z}$.*

Moreover, $|K_l|_\gamma = \frac{2^{q-1-l}}{2^q-1}\left(l + \frac{q}{2^q-1}\right)$, and if $E_l^-(\gamma) := \bigcup_{i<0} E_l^i(\gamma)$, $E_l^+(\gamma) := \bigcup_{i>0} E_l^i(\gamma)$, then $|E_l^+(\gamma)|_\gamma = \frac{s_1-\gamma}{2^l}\left(l + \frac{q}{1-2^{-q}}\right)$ and $|E_l^-(\gamma)|_\gamma = \frac{\gamma+1/2-s_q}{2^l}\left(l + \frac{q}{1-2^{-q}}\right)$.

DEFINITION 2.7. The *entrance time median* of T_γ is defined to be the unique[3] $\omega(\gamma) \in [\gamma, \gamma+1/2]$ such that

$$\int_\gamma^{\omega(\gamma)} e_\gamma = 1/2 = \int_{\omega(\gamma)}^{\gamma+1/2} e_\gamma \,.$$

A remarkable fact discovered in [**2**, Thm. 6.4, Cor. 6.6] is that, provided $\varrho(\gamma)$ is not one of the nine rotation numbers 0, $\pm 2/7$, $\pm 3/10$, $\pm 1/3$, $\pm 3/8$, the location of $\omega(\gamma)$ is rather stable:

THEOREM 2.8. *The entrance time median $\omega(\gamma)$ lies in $E_2(\gamma)$ if $\varrho(\gamma) \in (0, 2/7)$, in $E_{12}(\gamma)$ if $\varrho(\gamma) \in (2/7, 3/10)$, in $E_5(\gamma)$ if $\varrho(\gamma) \in (3/10, 1/3)$, in $E_4(\gamma)$ if $\varrho(\gamma) \in (1/3, 3/8)$, and in $E_1(\gamma)$ if $\varrho(\gamma) \in (3/8, 1/2]$.*

Moreover if T_γ has rational rotation number $\varrho(\gamma) = p/q$ (where $q \in \mathbb{N}$ and the integer $0 < p < q$ is coprime to q) then, more precisely, $\omega(\gamma)$ lies in $J_{q-2p} \subset E_2(\gamma)$ if $\varrho(\gamma) \in (0, 2/7)$, in $J_{4q-12p} \subset E_{12}(\gamma)$ if $\varrho(\gamma) \in (2/7, 3/10)$, in $J_{2q-5p} \subset E_5(\gamma)$ if $\varrho(\gamma) \in (3/10, 1/3)$, in $J_{2q-4p} \subset E_4(\gamma)$ if $\varrho(\gamma) \in (1/3, 3/8)$, and in $J_{q-p} \subset E_1(\gamma)$ if $\varrho(\gamma) \in (3/8, 1/2]$

3. Complete determination of the map $\omega : \mathbb{T} \to \mathbb{T}$

Theorem 2.8 localises $\omega(\gamma)$ to some extent, identifying the orbit partition piece $K_l = J_{\pi_{p,q}(l)}$ which contains it in the case when $\varrho(\gamma)$ is rational and not equal to one of the exceptional values 0, $\pm 2/7$, $\pm 3/10$, $\pm 1/3$, or $\pm 3/8$. Our aim in this section and the next is to go beyond this result, obtaining an *explicit* description of the entrance time median $\omega(\gamma)$, and one which is valid for all rotation numbers.

[3] Uniqueness is because e_γ is strictly positive on $\mathbb{T} \setminus F_\gamma = [\gamma, \gamma + 1/2]$.

In what follows it will be useful to consider those partition pieces which lie *to the left* of K_l, namely the collection of those $J_i = K_{\pi^{-1}_{p,q}(i)}$ for which $1 \le i < \pi_{p,q}(l)$, so for future use we introduce the notation

$$I_l(p/q) := \sum_{i=1}^{\pi_{p,q}(l)-1} |K_{\pi^{-1}_{p,q}(i)}|_\gamma = \frac{2^{q-1}}{2^q-1} \sum_{i=1}^{\pi_{p,q}(l)-1} \left(\pi^{-1}_{p,q}(i) + \frac{q}{2^q-1}\right) 2^{-\pi^{-1}_{p,q}(i)} \,,$$

where γ denotes *any* parameter value in the interval $\varrho^{-1}(p/q)$, noting that by Lemma 2.6 the quantity $I_l(p/q)$ is indeed independent of $\gamma \in \varrho^{-1}(p/q)$.

Notation 3.1. For $\gamma \in \varrho^{-1}(p/q)$, $1 \le l \le q-1$, $i \in \mathbb{Z}$, let $\epsilon^i_l(\gamma)$ denote the left endpoint of $E^i_l(\gamma)$, so that, by Lemma 2.6, $\epsilon^{i+1}_l(\gamma)$ is the right endpoint of $E^i_l(\gamma)$.

Lemma 3.2. *For a rational rotation number p/q, if $[\gamma_1, \gamma_2]$ is a sub-interval of $\varrho^{-1}(p/q)$, and there exist $1 \le l \le q-1$, $i \in \mathbb{Z}$, such that $\omega(\gamma) \in E^i_l(\gamma)$ for all $\gamma \in [\gamma_1, \gamma_2]$, then the restriction of ω to $[\gamma_1, \gamma_2]$ is affine increasing of slope $\frac{1-2^{-M}}{M}\frac{q}{1-2^{-q}}$, where $M = l + |i|q$.*

Proof. The median $\omega(\gamma)$ is defined by the equation

$$1/2 = \int_\gamma^{\omega(\gamma)} e_\gamma = |[\gamma, \omega(\gamma)]|_\gamma = |[\gamma, \epsilon^i_l(\gamma)]|_\gamma + |[\epsilon^i_l(\gamma), \omega(\gamma)]|_\gamma \,. \tag{2}$$

For each $\gamma \in [\gamma_1, \gamma_2]$, the level set $E_l(\gamma)$ is an interval (rather than a union of two disjoint intervals), and $E_l(\gamma)$ is mapped onto $E_0(\gamma) = F_\gamma = (\gamma + 1/2, \gamma)$ by T^l; hence there exists a dyadic rational constant $b \in \mathbb{T}$ such that $E_l(\gamma) = (2^{-l}\gamma + b, 2^{-l}\gamma + b + 2^{-(l+1)})$ for all $\gamma \in [\gamma_1, \gamma_2]$. Therefore $\epsilon^i_l(\gamma) = 2^{-M}\gamma + b$, so $|[\epsilon^i_l(\gamma), \omega(\gamma)]|_\gamma = M(\omega(\gamma) - 2^{-M}\gamma - b)$. Now $\gamma \mapsto |[\gamma, \epsilon^i_l(\gamma)]|_\gamma$ is affine of slope $\frac{M}{2^M} - \frac{q}{1-2^{-q}}\left(1 - 2^{-M}\right)$, by [**2**, Lem. 5.9], so the righthand side of (2) is of the form $M\omega(\gamma) - \frac{q}{1-2^{-q}}\left(1-2^{-M}\right)\gamma + c$, for some constant c, and therefore $\omega(\gamma) = \frac{1-2^{-M}}{M}\frac{q}{1-2^{-q}}\gamma + (1/2 - c)/M$. □

With more care, we are able to derive an explicit formula for the map ω on any interval $[\gamma_1, \gamma_2]$ satisfying the hypotheses of Lemma 3.2, in particular when $i = 0$ and $[\gamma_1, \gamma_2] = \varrho^{-1}(p/q)$:

Lemma 3.3. *For a rational rotation number p/q, if there exists $1 \le l \le q-1$ such that $\omega(\gamma) \in E_l(\gamma)$ for all $\gamma \in \varrho^{-1}(p/q)$, then*

$$\omega(\gamma) = \frac{1-2^{-l}}{l}\frac{q}{1-2^{-q}}\gamma + a_{p,q,l}$$

for $\gamma \in \varrho^{-1}[p/q]$, where $a_{p,q,l} = b_{p,q,l} + \frac{1}{l}\left(\frac{1}{2} - I_l(p/q) - \frac{s_1 q}{1-2^{-q}} + \frac{s_q - 1/2}{2^l}\left(l + \frac{q}{1-2^{-q}}\right)\right)$, and $b_{p,q,l}$ is such that the left endpoint of $E_l(\gamma)$ equals $2^{-l}\gamma + b_{p,q,l}$.

PROOF. Now $[\gamma,\omega(\gamma)] = \overline{E_0^+(\gamma)} \cup \left(\cup_{1\le i<\pi_{p,q}(l)} K_{\pi_{p,q}^{-1}(i)}\right) \cup \overline{E_l^-(\gamma)} \cup [2^{-l}\gamma + b_{p,q,l}, \omega(\gamma)]$, so $1/2 = \int_\gamma^{\omega(\gamma)} e_\gamma = |E_0^+(\gamma)|_\gamma + I_l(p/q) + |E_l^-(\gamma)|_\gamma + l\left(\omega(\gamma) - 2^{-l}\gamma - b_{p,q,l}\right)$. Thus $\omega(\gamma) = 2^{-l}\gamma + b_{p,q,l} + \frac{1}{l}\left(\frac12 - |E_0^+(\gamma)|_\gamma - I_l(p/q) - |E_l^-(\gamma)|_\gamma\right)$, so Lemma 2.6 gives

$$\omega(\gamma) = 2^{-l}\gamma + b_l + \frac{1}{l}\left(\frac12 - I_l(p/q) - (s_1-\gamma)\frac{q}{1-2^{-q}} - \frac{\gamma+1/2-s_q}{2^l}\left(l + \frac{q}{1-2^{-q}}\right)\right),$$

which, after rearrangement, becomes the required formula. □

Computaton of the coefficients $a_{p,q,l}$ and $b_{p,q,l}$ are illustrated by the following example:

EXAMPLE 3.4. Let $p/q = 2/5$. Then $\omega(\gamma) \in E_1(\gamma)$ for all $\gamma \in \varrho^{-1}(2/5)$, by Theorem 2.8. The left endpoint of $E_1(\gamma)$ is $\gamma/2 + 1/4$, hence $b_{2,5,1} = \gamma/2 + 1/4 - \gamma/2 = 1/4$. The permutation $\pi = \pi_{2,5}$ of $\{1,2,3,4\}$ is given by $\pi(1) = 3$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 2$, so $I_1(2/5) = \frac{2^{5-1}}{2^5-1}\sum_{i=1}^2 \left(\pi^{-1}(i) + \frac{5}{2^5-1}\right) 2^{-\pi^{-1}(i)} = \frac{16}{31}\left[\left(2+\frac{5}{31}\right)2^{-2} + \left(4 + \frac{5}{31}\right)2^{-4}\right] = \frac{397}{961}$. Therefore, $a_{2,5,1} = \frac14 + \frac12 - \frac{397}{961} - \frac{25}{31(1-2^{-5})} + \frac{9}{2\cdot 62}\left(1 + \frac{5}{1-2^{-5}}\right) = -\frac{3}{62}$, and thus $\omega(\gamma) = \frac{80}{31}\gamma - \frac{3}{62}$ for all $\gamma \in \varrho^{-1}(2/5) = [9/62, 5/31]$.

PROPOSITION 3.5. *If p/q is an exceptional rotation number* $0, \pm 2/7, \pm 3/10, \pm 1/3$, *or* $\pm 3/8$, *then there exists a parameter $\gamma_\infty(p/q) \in \varrho^{-1}(p/q)$ such that the entrance time median $\omega(\gamma_\infty(p/q))$ lies in the Sturmian orbit $\mathcal{O}_{p/q}$.*

PROOF. Clearly $\gamma_\infty(0) = -1/4$, and the entrance time median of $T_{-1/4}$ is the fixed point 0 (since $e_{-1/4}$ is an even function, cf. [**2**, Ex. 5.6]). By symmetry we only need consider exceptional rotation numbers p/q equal to 2/7, 3/10, 1/3 or 3/8. For these values, let $M(p/q)$ denote the constant value of $e_\gamma(\omega(\gamma))$ for those γ such that $\varrho(\gamma)$ is to the left of, and sufficiently close to, p/q (cf. Theorem 2.8); explicitly, $M(2/7) = 2$, $M(3/10) = 12$, $M(1/3) = 5$, and $M(3/8) = 4$. If $N = N(p/q) := M(p/q) \pmod q$ then the point in the orbit $\mathcal{O}_{p/q}$ which coincides with the entrance time median of some $\gamma \in \varrho^{-1}(p/q)$ is $s_{\pi_{p,q}(N)+1}$, and $\gamma = \gamma_\infty(p/q)$ is the solution to the equation $1/2 = \int_\gamma^{s_{\pi_{p,q}(N)+1}} e_\gamma$, or, more explicitly,

$$1/2 = |E_0^+(\gamma)|_\gamma + \sum_{i=1}^{\pi_{p,q}(N)} |K_{\pi_{p,q}^{-1}(i)}|_\gamma\,. \tag{3}$$

Precisely, $N(p/q)$ equals 4 when $p/q = 3/8$, and equals 2 when p/q is either 2/7, 3/10, or 1/3; thus $\omega(\gamma_\infty(2/7)) = s_4(2/7) = 34/127$, $\omega(\gamma_\infty(3/10)) = s_5(3/10) = 274/1023$, $\omega(\gamma_\infty(1/3)) = s_2(1/3) = 2/7$, and $\omega(\gamma_\infty(3/8)) = s_5(3/8) = 82/255$.

The equation (3) can be solved explicitly: $\gamma_\infty(2/7) = 16097/227584 \approx 0.0707$, $\gamma_\infty(3/10) = 1485761/20951040 \approx 0.0709$, $\gamma_\infty(1/3) = 11/112 \approx 0.0982$, and $\gamma_\infty(3/8) = 149977/1044480 \approx 0.1436$. □

REMARK 3.6. Symmetry considerations clearly give $\gamma_\infty(-3/8) = 1/2 - \gamma_\infty(3/8) \approx 0.3564$, $\gamma_\infty(-1/3) = 1/2 - \gamma_\infty(1/3) \approx 0.4018$, $\gamma_\infty(-3/10) = 1/2 - \gamma_\infty(3/10) \approx 0.4291$, and $\gamma_\infty(-2/7) = 1/2 - \gamma_\infty(2/7) \approx 0.4293$.

NOTATION 3.7. The nine values $\gamma_\infty(0)$, $\gamma_\infty(\pm 2/7)$, $\gamma_\infty(\pm 3/10)$, $\gamma_\infty(\pm 1/3)$, $\gamma_\infty(\pm 3/8)$ will be referred to as *critical parameters*.

The critical parameters define a finite partition of parameter space $\mathbb{T}$:

NOTATION 3.8. Define $\Gamma_2 := (\gamma_\infty(0), \gamma_\infty(2/7))$, $\Gamma_{12} := (\gamma_\infty(2/7), \gamma_\infty(3/10))$, $\Gamma_5 := (\gamma_\infty(3/10), \gamma_\infty(1/3))$, $\Gamma_4 := (\gamma_\infty(1/3), \gamma_\infty(3/8))$, $\Gamma_1 := (\gamma_\infty(3/8), \gamma_\infty(-3/8))$, $\Gamma_{-4} := (\gamma_\infty(-3/8), \gamma_\infty(-1/3))$, $\Gamma_{-5} := (\gamma_\infty(-1/3), \gamma_\infty(-3/10))$, $\Gamma_{-12} := (\gamma_\infty(-3/10), \gamma_\infty(-2/7))$, $\Gamma_{-2} := (\gamma_\infty(-2/7), \gamma_\infty(0))$. The collection $\{\Gamma_l\}$ of these 9 intervals will be called the *critical partition* of parameter space.

Define, for each $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, the integers $p_l^-, p_l^+ \in \mathbb{Z}$ and positive integers $q_l^-, q_l^+ \in \mathbb{N}$ by $\Gamma_l = (\gamma_\infty(p_l^-/q_l^-), \gamma_\infty(p_l^+/q_l^+))$, where $\gcd(p_l^-, q_l^-) = 1$ and $\gcd(p_l^+, q_l^+) = 1$.

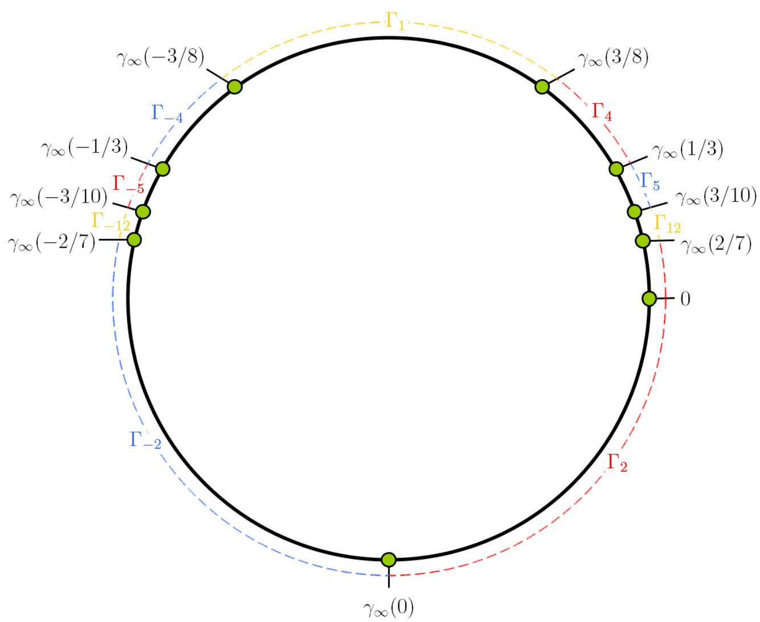

FIGURE 4. The critical partition $\{\Gamma_l\}$ of parameter space (cf. Notation 3.8).

It turns out to be important to further sub-divide the parameter intervals Γ_l; consideration of the following subintervals of each critical partition piece Γ_l is motivated by Propositions 3.11 and 3.12:

NOTATION 3.9. For all $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, define

$$\Delta_l^0 := \{\gamma \in \Gamma_l : \omega(\gamma) \in E_l(\gamma)\}.$$

We now define Δ_l^i for $i \in \mathbb{Z} \setminus \{0\}$. If $l = 1$ or ± 2 then for $i \in \mathbb{Z} \setminus \{0\}$, define

$$\Delta_l^i := \{\gamma \in \Gamma_l : \omega(\gamma) \in E_l^i(\gamma)\}\,. \tag{4}$$

If $l = \pm 12$ then for $i < 0$ define

$$\Delta_l^i := \{\gamma \in \Gamma_l : \omega(\gamma) \in E_{l-q(\gamma)}^{i-1}(\gamma)\}\,, \tag{5}$$

and for $i > 0$ define

$$\Delta_l^i := \{\gamma \in \Gamma_l : \omega(\gamma) \in E_{l-q(\gamma)}^{i+1}(\gamma)\}\,. \tag{6}$$

If $l = 5$ or -4 then for $i < 0$ define Δ_l^i by (4), and for $i > 0$ define Δ_l^i by (6). If $l = 4$ or -5 then for $i < 0$ define Δ_l^i by (5), and for $i > 0$ define Δ_l^i by (4).

By Theorem 2.8, if $i \neq 0$ then each $\gamma \in \Delta_l^i$ has rotation number equal to $p_l^\pm/q_l^\pm$. This leads to the following more explicit descriptions of Δ_l^i (we list the cases $l \in \{2, 12, 5, 4, 1\}$; for reasons of symmetry these are the only l explicitly used in the following Proposition 3.11):

$$\begin{aligned}
\Delta_2^i &= \{\gamma \in \Gamma_2 : \omega(\gamma) \in E_2^i(\gamma)\} \quad \text{for } i \in \mathbb{Z}\,,\\
\Delta_1^i &= \{\gamma \in \Gamma_1 : \omega(\gamma) \in E_1^i(\gamma)\} \quad \text{for } i \in \mathbb{Z}\,,\\
\Delta_{12}^i &= \{\gamma \in \Gamma_{12} : \omega(\gamma) \in E_5^{i-1}(\gamma)\} \quad \text{for } i < 0\,,\\
\Delta_{12}^i &= \{\gamma \in \Gamma_{12} : \omega(\gamma) \in E_2^{i+1}(\gamma)\} \quad \text{for } i > 0\,,\\
\Delta_5^i &= \{\gamma \in \Gamma_5 : \omega(\gamma) \in E_5^i(\gamma)\} \quad \text{for } i < 0\,,\\
\Delta_5^i &= \{\gamma \in \Gamma_5 : \omega(\gamma) \in E_2^{i+1}(\gamma)\} \quad \text{for } i > 0\,,\\
\Delta_4^i &= \{\gamma \in \Gamma_4 : \omega(\gamma) \in E_1^{i-1}(\gamma)\} \quad \text{for } i < 0\,,\\
\Delta_4^i &= \{\gamma \in \Gamma_4 : \omega(\gamma) \in E_4^i(\gamma)\} \quad \text{for } i > 0\,.
\end{aligned}$$

To prove Proposition 3.11 we first require the following technical result from [**2**, Lem. 6.1]:

Lemma 3.10. *For rational rotation number p/q, and $1 \leq l \leq q-1$, $i \in \mathbb{Z}$, suppose that $\gamma_1, \gamma_2 \in \varrho^{-1}(p/q)$ satisfy $\omega(\gamma_1) \in \overline{E_l^i(\gamma_1)}$ and $\omega(\gamma_2) \in \overline{E_l^i(\gamma_2)}$. Then $\omega(\gamma) \in E_l^i(\gamma)$ for all $\gamma \in (\gamma_1, \gamma_2)$.*

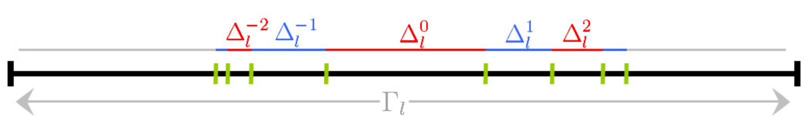

Figure 5. The partition $\{\Delta_l^i\}_{i\in\mathbb{Z}}$ of the critical partition piece Γ_l (cf. Prop. 3.11).

Proposition 3.11. *For each $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, $\{\Delta_l^i\}_{i\in\mathbb{Z}}$ is a collection of pairwise disjoint sub-intervals of $\mathbb{T}$, with $\Delta_l^i \prec \Delta_l^{i+1}$ for all $i \in \mathbb{Z}$, and $\Gamma_l = \cup_{i\in\mathbb{Z}}\Delta_l^i$.*

PROOF. By symmetry it suffices to consider the case $l \in \{2, 12, 5, 4, 1\}$. For such l we make the following claims: There exists $\gamma_l^0 \in \Gamma_l$ such that $\omega(\gamma_l^0)$ is the left endpoint of $E_l(\gamma_l^0)$, and $\gamma_l^1 \in \Gamma_l$ such that $\omega(\gamma_l^1)$ is the right endpoint of $E_l(\gamma_l^1)$. If $l = 1$ or 2, for each $i \in \mathbb{Z} \setminus \{0\}$ there exists $\gamma_l^i \in \Gamma_l$ with $\omega(\gamma_l^i) = \epsilon_l^i(\gamma_l^i)$. For $i < 0$ there exist $\gamma_{12}^i \in \Gamma_{12}$ with $\omega(\gamma_{12}^i) = \epsilon_5^{i-1}(\gamma_{12}^i)$, and $\gamma_5^i \in \Gamma_5$ with $\omega(\gamma_5^i) = \epsilon_5^i(\gamma_5^i)$, and $\gamma_4^i \in \Gamma_4$ with $\omega(\gamma_4^i) = \epsilon_1^{i-1}(\gamma_4^i)$. For $i > 0$ there exist $\gamma_{12}^i \in \Gamma_{12}$ with $\omega(\gamma_{12}^i) = \epsilon_2^{i+1}(\gamma_{12}^i)$, and $\gamma_5^i \in \Gamma_5$ with $\omega(\gamma_5^i) = \epsilon_2^{i+1}(\gamma_5^i)$, and $\gamma_4^i \in \Gamma_4$ with $\omega(\gamma_4^i) = \epsilon_4^i(\gamma_4^i)$. Moreover, $\gamma_l^i < \gamma_l^{i+1}$ for all $i \in \mathbb{Z}$ (where $<$ denotes the ordering in Γ_l), and $\lim_{i\to\pm\infty} \gamma_l^i = \gamma_\infty(p_l^\pm/q_l^\pm)$.

The proofs of the above claims involve separate calculations for each l. Here we restrict ourselves to indicating the results in the case $l = 2$, $i \leq 0$, the proofs in other cases being very similar. The parameter $\gamma_{\max}(0) = [0, 1/2]$ has entrance time median $1/8$, which is precisely the left endpoint $\epsilon_2^0(\gamma_{\max}(0))$ of $E_2^0(\gamma_{\max}(0))$ (cf. Theorem 2.8); thus $\gamma_2^0 = \gamma_{\max}(0)$. More generally, for $i \leq 0$ the parameter γ_2^i can be computed to be $\frac{4-i}{4(2^{3-i}+i-4)} - 1/4$.

Now define $\hat{\Delta}_l^i := [\gamma_l^i, \gamma_l^{i+1})$ for $i > 0$, $\hat{\Delta}_l^i := (\gamma_l^i, \gamma_l^{i+1}]$ for $i < 0$, and $\hat{\Delta}_l^0 := (\gamma_l^0, \gamma_l^1)$. Clearly these sub-intervals of $\mathbb{T}$ are pairwise disjoint, with $\hat{\Delta}_l^i \prec \hat{\Delta}_l^{i+1}$ for all $i \in \mathbb{Z}$, and $\Gamma_l = \cup_{i\in\mathbb{Z}} \hat{\Delta}_l^i$. To complete the proof, it now suffices to check that $\hat{\Delta}_l^i = \Delta_l^i$ for each i.

We provide full details that $\hat{\Delta}_l^i = \Delta_l^i$ in the case $l = 2$; the proofs for other l are similar, involving appropriate modifications in cases where the definition of Δ_l^i differs from (4). If $i \in \mathbb{Z} \setminus \{0\}$ then $\omega(\gamma_2^i) = \epsilon_2^i(\gamma_2^i)$ is the left endpoint of $E_2^i(\gamma_2^i)$. Also, $\omega(\gamma_2^{i+1}) = \epsilon_2^{i+1}(\gamma_2^{i+1})$ is the right endpoint of $E_2^i(\gamma_2^{i+1})$. If $i < 0$ then $\gamma_2^i, \gamma_2^{i+1} \in \varrho^{-1}(0)$, and if $i > 0$ then $\gamma_2^i, \gamma_2^{i+1} \in \varrho^{-1}(2/7)$; in both cases Lemma 3.10 implies that $\omega(\gamma) \in E_2^i(\gamma)$ for all $\gamma \in \hat{\Delta}_2^i$. That is,

$$\hat{\Delta}_2^i \subset \Delta_2^i \quad \text{for } i \in \mathbb{Z} \setminus \{0\}\,. \tag{7}$$

We now wish to show that $\hat{\Delta}_2^0 \subset \Delta_2^0$. Since

$$\hat{\Delta}_2^0 = (\gamma_2^0, \gamma_2^1) = (\gamma_{\max}(0), \gamma_2^1) = (\gamma_{\max}(0), \gamma_{\min}(2/7)) \cup [\gamma_{\min}(2/7), \gamma_2^1)\,,$$

and $\omega(\gamma) \in E_2(\gamma)$ for $\gamma \in (\gamma_{\max}(0), \gamma_{\min}(2/7))$ by Theorem 2.8, it only remains to show that $\omega(\gamma) \in E_2(\gamma)$ for $\gamma \in [\gamma_{\min}(2/7), \gamma_2^1)$. Now $\omega(\gamma_2^1) = \epsilon_2^1(\gamma_2^1)$ is the right endpoint of $E_2^0(\gamma_2^1)$, and by Theorem 2.8, $\omega(\gamma_{\min}(2/7))$ lies in $E_2(\gamma_{\min}(2/7)) = E_2^0(\gamma_{\min}(2/7))$. Lemma 3.10 therefore implies that $\omega(\gamma) \in E_2^0(\gamma)$ for all $\gamma \in [\gamma_{\min}(2/7), \gamma_2^1)$, so indeed

$$\hat{\Delta}_2^0 \subset \Delta_2^0\,. \tag{8}$$

Since $\{\hat{\Delta}_2^i\}_{i\in\mathbb{Z}}$ is a partition of Γ_2, we see that the inclusions in (7) and (8) are in fact equalities, as required. □

We are now in a position to describe $\omega : \mathbb{T} \to \mathbb{T}$ on all intervals $\varrho^{-1}(p/q)$:

PROPOSITION 3.12. *The map $\omega : \mathbb{T} \to \mathbb{T}$ is countably piecewise affine: for $i \in \mathbb{Z}$ and $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, the map ω is affine increasing on the interval Δ_l^i.*

More precisely, for $i \neq 0$, the slope of ω on Δ_l^i equals

$$\frac{q_l^-}{1-2^{-q_l^-}}\frac{1-2^{-(|l|-iq_l^-)}}{|l|-iq_l^-} \quad \text{if } i<0\,,$$

and

$$\frac{q_l^+}{1-2^{-q_l^+}}\frac{1-2^{-(|l|+iq_l^+)}}{|l|+iq_l^+} \quad \text{if } i>0\,.$$

The slope of ω on $\Delta_l^0 \cap \varrho^{-1}(p_l^-/q_l^-) = (\gamma_l^0, \gamma_{\max}(p_l^-/q_l^-)]$ is $\frac{q_l^-}{1-2^{-q_l^-}}\frac{1-2^{-|l|}}{|l|}$, and on $\Delta_l^0 \cap \varrho^{-1}[p_l^+/q_l^+] = [\gamma_{\min}(p_l^+/q_l^+), \gamma_l^1)$ is $\frac{q_l^+}{1-2^{-q_l^+}}\frac{1-2^{-|l|}}{|l|}$.

If p/q is a non-critical rotation number then $\varrho^{-1}(p/q)$ lies in Δ_l^0 for some $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, and the slope of ω on $\varrho^{-1}(p/q)$ is $\frac{q}{1-2^{-q}}\frac{1-2^{-|l|}}{|l|}$.

Proof. This follows by combining Theorem 2.8 and Lemma 3.2 with Proposition 3.11. □

We can now prove, as announced in Section 1, that the countably piecewise affine map $\omega : \mathbb{T} \to \mathbb{T}$ is a degree-1 homeomorphism with *critical points* (i.e. points $\gamma \in \mathbb{T}$ at which ω is differentiable, with derivative equal to zero):

Theorem 3.13. *The map $\omega : \mathbb{T} \to \mathbb{T}$ is an increasing homeomorphism which is countably piecewise affine. Its critical points are precisely the 9 critical parameters $\gamma_\infty(0)$, $\gamma_\infty(\pm 2/7)$, $\gamma_\infty(\pm 3/10)$, $\gamma_\infty(\pm 1/3)$, and $\gamma_\infty(\pm 3/8)$, and it has cusps at those γ for which $\varrho(\gamma)$ is irrational.*

Proof. First we show that critical parameters are precisely the critical points of $\omega : \mathbb{T} \to \mathbb{T}$. By Proposition 3.12, for some $l \in \{\pm 2, \pm 12, \pm 5, \pm 4, 1\}$, the left-hand (respectively righthand) derivative of ω at any critical parameter $\gamma_\infty(p/q)$ is $\lim_{i\to\infty} \frac{q}{1-2^{-q}}\frac{1-2^{-(|l|+iq)}}{|l|+iq} = 0$ (respectively $\lim_{i\to-\infty} \frac{q}{1-2^{-q}}\frac{1-2^{-(|l|-iq)}}{|l|-iq} = 0$), so $\gamma_\infty(p/q)$ is a critical point of ω.

To show that ω has no other critical points, note firstly that for non-exceptional rational rotation numbers p/q, Lemma 3.3 implies that the slope of ω on the interval $\varrho^{-1}(p/q)$ equals $\frac{1-2^{-l}}{l}\frac{q}{1-2^{-q}} \geq \frac{1-2^{-12}}{12}\frac{2}{1-2^{-2}} > 0$, since $l \leq 12$ and $q \geq 2$. Secondly, ω is not locally Lipschitz at any γ_0 for which $\varrho(\gamma_0) \notin \mathbb{Q}$, since by Lemma 3.3, γ_0 can be approximated by parameters γ for which the slope of ω at γ equals $K\frac{q}{1-2^{-q}}$, for q arbitrarily large (and $K = \frac{1-2^{-l}}{l}$, where l equals either 1, 2, 4, 5, or 12).

To see that $\omega : \mathbb{T} \to \mathbb{T}$ is an increasing homeomorphism note that $\mathbb{T}^2 \ni (\gamma, \omega) \mapsto \int_\gamma^\omega e_\gamma$ is continuous, thus $\gamma \mapsto \omega(\gamma)$ has closed graph, and is therefore continuous. For each rational $p/q \in \mathbb{T}$, the map ω is strictly increasing on the interval $[\gamma_{\min}(p/q), \gamma_{\max}(p/q)]$, by Proposition 3.12, and the union of these intervals is dense in $\mathbb{T}$, so continuity of ω implies that it is strictly increasing on all of $\mathbb{T}$, i.e. is a degree-one homeomorphism. □

4. Computing the endpoints of the image intervals $\omega(\varrho^{-1}(p/q))$

In this final section we derive formulae for the endpoints $\omega_{\min}(p/q) := \omega(\gamma_{\min}(p/q))$ and $\omega_{\max}(p/q) := \omega(\gamma_{\max}(p/q))$ of the image intervals $\Omega(p/q) := \omega(\varrho^{-1}(p/q))$. Our first formula is rather abstract, relying on an assumption about the first entrance time of $\omega(\gamma)$:

Lemma 4.1. *For rational rotation number p/q, if there exists $1 \le l \le q-1$ such that $\omega(\gamma) \in E_l(\gamma)$ for all $\gamma \in \varrho^{-1}(p/q)$, then*

$$\omega_{\min}(p/q) = s_{\pi_{p,q}(l)} + \frac{1}{l}\left(\frac{1}{2} - I_l(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}\right).$$

Proof. Substitute $\gamma = \gamma_{\min}(p/q) = s_q - 1/2$ into the formula

$$\omega(\gamma) = 2^{-l}\gamma + b_l + \frac{1}{l}\left(\frac{1}{2} - I_l(p/q) - (s_1-\gamma)\frac{q}{1-2^{-q}} - \frac{\gamma + 1/2 - s_q}{2^l}\left(l + \frac{q}{1-2^{-q}}\right)\right)$$

derived in the proof of Lemma 3.3. Noting that the final term on the righthand side vanishes, that $s_1 - \gamma = 1/(2(2^q-1))$, and that the left endpoint $2^{-l}\gamma + b_l$ of $E_l(\gamma)$ equals the left endpoint $s_{\pi_{p,q}(l)}$ of $K_l = J_{\pi_{p,q}(l)}$, the required formula follows. □

Remark 4.2. The hypotheses of Lemma 4.1 also yield, via an analogous proof, the formula $\omega_{\max}(p/q) = s_{\pi_{p,q}(l)} + \frac{1}{l}\left(\frac{1}{2} - I_l(p/q) - \frac{2^{q-1}q}{2^l(2^q-1)^2}\right)$, and hence the following formula for the *length* of the interval $[\omega_{\min}(p/q), \omega_{\min}(p/q)]$:

$$\omega_{\max}(p/q) - \omega_{\min}(p/q) = \frac{1-2^{-l}}{l}\frac{2^{q-1}q}{(2^q-1)^2}. \tag{9}$$

The following theorem renders Lemma 4.1 concrete, giving formulae for $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ when p/q is non-exceptional:

Theorem 4.3. *Let $s_1 < \ldots < s_q$ be the Sturmian orbit of rotation number p/q.*

(i) *If $0 < \frac{p}{q} < \frac{2}{7}$ then $\omega_{\min}(p/q) = s_{q-2p} + \frac{1}{2}\left(\frac{1}{2} - I_2(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}\right)$ and $\omega_{\max}(p/q) = \omega_{\min}(p/q) + \frac{3}{16}\frac{2^q q}{(2^q-1)^2}$, where $I_2(p/q) = \frac{2^{q-1}}{2^q-1}\sum_{i=1}^{q-2p-1}\left(\pi_{p,q}^{-1}(i) + \frac{q}{2^q-1}\right)2^{-\pi_{p,q}^{-1}(i)}$.*

(ii) *If $\frac{2}{7} < \frac{p}{q} < \frac{3}{10}$ then $\omega_{\min}(p/q) = s_{4q-12p} + \frac{1}{12}\left(\frac{1}{2} - I_{12}(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}\right)$, and $\omega_{\max}(p/q) = \omega_{\min}(p/q) + \frac{1365}{32768}\frac{2^q q}{(2^q-1)^2}$, where $I_{12}(p/q) = \frac{2^{q-1}}{2^q-1}\sum_{i=1}^{4q-12p-1}\left(\pi_{p,q}^{-1}(i) + \frac{q}{2^q-1}\right)2^{-\pi_{p,q}^{-1}(i)}$.*

(iii) *If $\frac{3}{10} < \frac{p}{q} < \frac{1}{3}$ then $\omega_{\min}(p/q) = s_{2q-5p} + \frac{1}{5}\left(\frac{1}{2} - I_5(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}\right)$ and $\omega_{\max}(p/q) = \omega_{\min}(p/q) + \frac{31}{320}\frac{2^q q}{(2^q-1)^2}$, where $I_5(p/q) = \frac{2^{q-1}}{2^q-1}\sum_{i=1}^{2q-5p-1}\left(\pi_{p,q}^{-1}(i) + \frac{q}{2^q-1}\right)2^{-\pi_{p,q}^{-1}(i)}$.*

(iv) *If $\frac{1}{3} < \frac{p}{q} < \frac{3}{8}$ then $\omega_{\min}(p/q) = s_{2q-4p} + \frac{1}{4}\left(\frac{1}{2} - I_4(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}\right)$ and $\omega_{\max}(p/q) = \omega_{\min}(p/q) + \frac{15}{128}\frac{2^q q}{(2^q-1)^2}$, where $I_4(p/q) = \frac{2^{q-1}}{2^q-1}\sum_{i=1}^{2q-4p-1}\left(\pi_{p,q}^{-1}(i) + \frac{q}{2^q-1}\right)2^{-\pi_{p,q}^{-1}(i)}$.*

(v) *If $\frac{3}{8} < \frac{p}{q} \le \frac{1}{2}$ then $\omega_{\min}(p/q) = s_{q-p} + \frac{1}{2} - I_1(p/q) - \frac{2^{q-1}q}{(2^q-1)^2}$ and $\omega_{\max}(p/q) = \omega_{\min}(p/q) + \frac{1}{4}\frac{2^q q}{(2^q-1)^2}$, where $I_1(p/q) = \frac{2^{q-1}}{2^q-1}\sum_{i=1}^{q-p-1}\left(\pi_{p,q}^{-1}(i) + \frac{q}{2^q-1}\right)2^{-\pi_{p,q}^{-1}(i)}$.*

Proof. By Theorem 2.8 we know that $\omega(\gamma) \in E_l(\gamma)$ for some appropriate $l \in \{2, 12, 5, 4, 1\}$. Substituting this l into the formulae of Lemma 4.1 and Remark 4.2 completes the proof. □

Corollary 4.4. *For $q \geq 4$,*

$$\omega_{\min}(1/q) = \frac{2^{2q-1} + (2+3q)2^{q-2} - 1}{4(2^q-1)^2} \quad , \quad \omega_{\max}(1/q) = \frac{2^{2q-1} + (1+3q)2^{q-1} - 1}{4(2^q-1)^2} .$$

Proof. This follows easily from Theorem 4.3 (i). □

Remark 4.5. The Sturmian points $s_1 < \ldots < s_q$ appearing in Theorem 4.3 are in practice easily computed. For example $s_1 = \frac{2^q}{2(2^q-1)} \sum_{i=1}^{p} 2^{-\pi_{p,q}^{-1}(i)}$, $s_q = s_1 + \frac{1}{2} - \frac{1}{2(2^q-1)}$, and the other s_j appearing in Theorem 4.3 can be deduced by iteration.

Example 4.6. Let $p/q = 2/5$. Using Example 3.4, $I_1(2/5) = 397/961$. Now, the Sturmian orbit is $\frac{5}{31} < \frac{9}{31} < \frac{10}{31} < \frac{18}{31} < \frac{20}{31}$, so $s_{q-p} = s_3 = \frac{10}{31}$, and therefore $\omega_{\min}(2/5) = \frac{10}{31} + \frac{1}{2} - \frac{397}{961} - \frac{2^{5-1}5}{(2^5-1)^2} = \frac{627}{1922} \approx 0.3262$ and $\omega_{\max}(2/5) = \frac{627}{1922} + \frac{1}{4}\frac{2^5 5}{(2^5-1)^2} = \frac{707}{1922} \approx 0.3678$.

Theorem 4.3 gives the values $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ for all rationals p/q except for the exceptional rotation numbers 0, 2/7, 3/10, 1/3 and 3/8. The following theorem lists $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ for these exceptional rotation numbers.

Theorem 4.7.
(i) $\omega_{\min}(0) = -\frac{1}{8} = -0.125$ *and* $\omega_{\max}(0) = \frac{1}{8} = 0.125$,
(ii) $\omega_{\min}(2/7) = \frac{16631}{64516} \approx 0.257781$ *and* $\omega_{\max}(2/7) = \frac{3317209}{12387072} \approx 0.267796$,
(iii) $\omega_{\min}(3/10) = \frac{1494731}{5581488} \approx 0.267802$ *and* $\omega_{\max}(3/10) = \frac{90709}{337590} \approx 0.268696$.
(iv) $\omega_{\min}(1/3) = \frac{527}{1960} \approx 0.268878$ *and* $\omega_{\max}(1/3) = \frac{251}{784} \approx 0.320153$.
(v) $\omega_{\min}(3/8) = \frac{11129}{34680} \approx 0.320905$ *and* $\omega_{\max}(3/8) = \frac{42371}{130050} \approx 0.325805$.

Proof. The derivation of these values requires ad hoc calculation in each case. Here we indicate the proof that $\omega_{\min}(1/3) = 527/1960$; the proofs of the other cases are similar.

Now $\omega_{\min}(1/3)$ is the entrance time median for parameter $\gamma = \gamma_{\min}(1/3) = 1/14$, and by Theorem 2.8, $\omega_{\min}(1/3) \in E_5(\gamma) = E_2^1(\gamma)$. Now $\pi_{1,3}(2) = 1$, so $E_2^1(\gamma) \subset J_1 = [s_1, s_2] = [1/7, 2/7]$, therefore $\omega_{\min}(1/3)$ is defined by the equation $\frac{1}{2} = |[1/14, 1/7]|_\gamma + |E_2(\gamma)|_\gamma + |[\epsilon_2^1(\gamma), \omega_{\min}(1/3)]|_\gamma$. That is, $\frac{1}{2} = 3\frac{2^3}{2^3-1}\lambda_3 + 1/4 + 5\left(\omega_{\min}(1/3) - \frac{1/14+1}{4}\right)$, and the result follows. □

Remark 4.8. In Example 4.6 and Theorem 4.7, we have explicitly derived the values $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ of certain rational rotation numbers p/q. We now take the opportunity to explicitly catalogue some further values, using the formulae of Theorem 4.3. Specifically, we compute $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ for all remaining p/q in the Farey level $\frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{2}{9} < \frac{1}{4} < \frac{3}{11} < \frac{2}{7} < \frac{3}{10} < \frac{1}{3} < \frac{4}{11} < \frac{3}{8} < \frac{5}{13} < \frac{2}{5} < \frac{5}{12} < \frac{3}{7} < \frac{4}{9} < \frac{1}{2}$, namely:

ϱ	$\omega_{\min}(\varrho)$	$\omega_{\max}(\varrho)$
1/6	$263/1764 \approx 0.149092$	$295/1764 \approx 0.167233$
1/5	$647/3844 \approx 0.16831$	$767/3844 \approx 0.199541$
2/9	$208583/1044484 \approx 0.19969$	$212039/1044484 \approx 0.203008$
1/4	$61/300 \approx 0.20333$	$77/300 \approx 0.25666$
3/11	$4303511/16760836 \approx 0.25675$	$4320407/16760836 \approx 0.257768$
4/11	$10736023/33521672 \approx 0.320271$	$10757143/33521672 \approx 0.3209011$
5/13	$43718531/134184962 \approx 0.325808$	$43771779/134184962 \approx 0.326205$
5/12	$4112657/11179350 \approx 0.36788$	$4120849/11179350 \approx 0.368613$
3/7	$11891/32258 \approx 0.36862$	$12339/32258 \approx 0.38251$
4/9	$199795/522242 \approx 0.382571$	$202099/522242 \approx 0.386983$
1/2	$7/18 \approx 0.388888$	$11/18 \approx 0.611111$

Knowledge of all the endpoints $\omega_{\min}(p/q)$ and $\omega_{\max}(p/q)$ allowed us to graph the map $\omega : \mathbb{T} \to \mathbb{T}$ (see Figure 3), and suggested the following result:

PROPOSITION 4.9. *There are only two parameters γ for which the entrance time median $\omega(\gamma)$ equals the centre $\gamma + 1/4$ of the flat spot complement $\mathbb{T} \setminus F_\gamma$, namely $\gamma = \pm 1/4$.*

PROOF. It is clear that if γ equals $\gamma_\infty(0) = -1/4$, or $1/4$, then the entrance time median of T_γ equals $\gamma + 1/4$, since in these cases the entrance time function e_γ is easily seen to be even (cf. Examples 5.6 and 5.8 in [**2**]).

Now define $\delta(\gamma) := \gamma + 1/4$. If $\delta(\gamma)$ lies in $(0, 1/2)$ then so does $\omega(\gamma)$, and in this case we claim that $\omega(\gamma) < \delta(\gamma)$ (where $<$ denotes the ordering on $(0, 1/2)$); since ω and δ are both odd functions, it follows that $\omega(\gamma) > \delta(\gamma)$ when $\delta(\gamma) \in (1/2, 1)$. To prove the claim, first recall that the restriction of ω to $(\gamma_\infty(0), \gamma_{\max}(0)]$ is countably piecewise affine, and the slope on any affine piece equals $2M^{-1}(1-2^{-M})$ for some integer $M \geq 2$, by Proposition 3.12. In particular, all of these slopes are $\leq 3/4 < 1$, so $\gamma \mapsto \omega(\gamma) - \delta(\gamma)$ is strictly decreasing on $(\gamma_\infty(0), \gamma_{\max}(0)]$. Secondly, if $\varrho(\gamma) \in (0, 3/8)$ then Proposition 3.11 implies that $\omega(\gamma)$ lies in either K_2, K_{12}, K_5, or K_4. Each of these partition pieces is to the left of K_1 (in the ordering on $\mathbb{T} \setminus F_\gamma$), hence to the left of $E_1(\gamma) \subset K_1$. Thus $\omega(\gamma)$ is to the left of $E_1(\gamma)$. But $E_1(\gamma)$ is a length-1/4 sub-interval of the length-1/2 interval $\mathbb{T} \setminus F_\gamma$, so must contain its centre $\delta(\gamma)$, and therefore $\omega(\gamma) < \delta(\gamma)$. Thirdly, if $\varrho(\gamma) = 3/8$ then $\omega(\gamma) \leq \omega(\gamma_{\max}(3/8)) = \frac{42371}{130050} < \frac{73}{510} + \frac{1}{4} = \delta(\gamma_{\min}(3/8)) \leq \delta(\gamma)$, by Theorem 4.7, and since both $\omega(\gamma)$ and $\delta(\gamma)$ are increasing with γ. Finally, if $\varrho(\gamma) \in (3/8, 1/2]$ then $\omega(\gamma) \in E_1(\gamma)$, by Theorem 2.8. By Lemma 3.3, ω is countably piecewise affine on $(3/8, 1/2]$, and its slope on any affine piece is $\frac{1}{2}\frac{q}{1-2^{-q}}$, where $q \geq 2$ is the denominator of the rotation number of the corresponding Sturmian orbit. In particular, each of these slopes is $\geq \frac{4}{3} > 1$, so $\gamma \mapsto \omega(\gamma) - \delta(\gamma)$ is strictly increasing on $(3/8, 1/2]$, its only zero being at $\gamma = 1/4$, and therefore $\omega(\gamma) < \delta(\gamma)$ for $\gamma \in (\gamma_{\max}(3/8), 1/4)$. □

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