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Maximum hitting frequency and fastest mean return time

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Abstract

Given a dynamical system and a subset A of its phase space, we consider those orbits which, on average, spend as long as possible in A. This largest possible average is the *maximum hitting frequency*. We study the variation of the (orbits of) maximum hitting frequency as a function of A.

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1. Introduction

Consider a dynamical system (a map T or flow φ_t) on a set X. For a given subset A of X, we shall consider the following question:

Which orbits spend most time in A?

That is, we are interested in those orbits whose *mean sojourn time* in A is as large as possible. The mean sojourn time for the orbit of x is defined as

$$\lim_{n\to\infty} \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\}$$

in discrete time and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_A(\phi_t(x))\, \mathrm{d}t$$

in continuous time, provided the limit exists. For discrete systems this mean sojourn time is the frequency with which the orbit of x *hits* (i.e. lands in) the set A, so we prefer to call it a *hitting frequency*[1]. The questions we shall address are: *Which orbits hit A most frequently? What is this maximum hitting frequency? How do these data vary with A?*

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[1] This terminology is rather less appropriate if time is continuous, but we retain it because this paper is mainly concerned with discrete time systems.

Of course, these questions are of little interest if there exist orbits which stay in A for all time: in that case, the maximum hitting frequency is 1, and the orbits with maximum hitting frequency are exactly those which remain in A. So our questions are complementary to those concerning so-called *open* dynamical systems (see, e.g., [CMT, De, GS, Lind, LiMa, PiYo, Sid, Urb2]), which consist precisely of those orbits which remain in A for all time. More precisely, an open system is the restriction of the original map (or flow) to the invariant set $\bigcap_n T^{-n}(A)$ (or $\bigcap_t \varphi_{-t}(A)$); the complement $X\setminus A$ is regarded as a *hole* in X, and any orbit which enters the hole is deemed to have *escaped* and is no longer part of the system. Clearly such systems are only of interest if the set of surviving orbits is non-empty.

The problem of maximum hitting frequencies may be regarded as a moderation of the problem of open systems, insofar as an orbit landing in $X\setminus A$ does not pay the ultimate price of disappearing, but instead is penalized in some fixed finite way. For both problems it is most natural to consider subsets A which are geometrically simple, for example connected sets with non-empty interior.

An alternative interpretation of hitting frequencies is in terms of *mean return times*. If an orbit hits the set A with maximum possible frequency, then its mean return time to A must be smaller than for all other orbits. Indeed, if it hits A with frequency α then its mean return time to A will be α^{-1}. This suggests the complementary notion of *fastest mean return time*, which will be explored in section 3.

Having defined the notion of maximum hitting frequency, a natural problem is to describe the way in which it varies as a function of the set A. To give a flavour of the type of results we might expect, consider the following concrete example. Let $T : [0, 1] \to [0, 1]$ be the quadratic map $T(x) = ax(1-x)$ at the Ulam–von Neumann parameter value $a = 4$ (cf [UvN]). Let $A = A_l = [(1-l)/2, (1+l)/2]$ be the closed interval of length l centred at the point $1/2$, and let $\alpha(l)$ be its maximum hitting frequency. We wish to study the way in which $\alpha(l)$ varies with l. If l is sufficiently large, then there are orbits which remain in A_l for all time, so that $\alpha(l) = 1$. Indeed, if $l \geqslant 1/2$ then the fixed point at $3/4$ lies in A_l, so $\alpha(l) = 1$ for $1/2 \leqslant l \leqslant 1$. On the other hand $\alpha(0) = 0$: no orbit visits $A_0 = \{1/2\}$ with positive frequency, since the point $1/2$ is not periodic. So $\alpha : [0, 1] \to [0, 1]$ is a non-decreasing function, increasing from the value $\alpha(0) = 0$ to the value $\alpha(1) = 1$. It will be shown (theorem 3) that in fact $\alpha(l)$ is a discontinuous function of l, only taking on the values $1/n$, for $n \geqslant 1$ an integer (see figure 1). Its discontinuities are at the points $l_n = \sin(\pi/2(2^n + 1))$, $n \geqslant 1$.

Of course, the Ulam–von Neumann map is a standard example of a *chaotic* dynamical system; in particular, it has a rich diversity of orbit types. This ensures a diversity of hitting frequencies associated with a typical set A, so that the problem of determining the (orbits with) maximum hitting frequency is a non-trivial one. For a general dynamical system, this problem is non-trivial provided there is more than one invariant probability measure, and it is most interesting when there are many such measures. This is the case for the one-dimensional expanding maps, which will be considered in sections 5 and 6, and more generally for any system enjoying some hyperbolicity. Various mechanical systems fall into this category, for example the geodesic flow on a compact surface of negative curvature (see [KH, chapter 17.5–6]), or the billiard flow on a table with dispersive boundary (see [Sin, Tab, chapter 5]). For such systems, where the phase space is the unit tangent bundle SM of some manifold M, rather than M itself, a potentially fruitful modification of our initial problem is to choose A as a subset of M, rather than of SM, and consider those orbits which maximize the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x, v))\, \mathrm{d}t, \qquad (1)$$

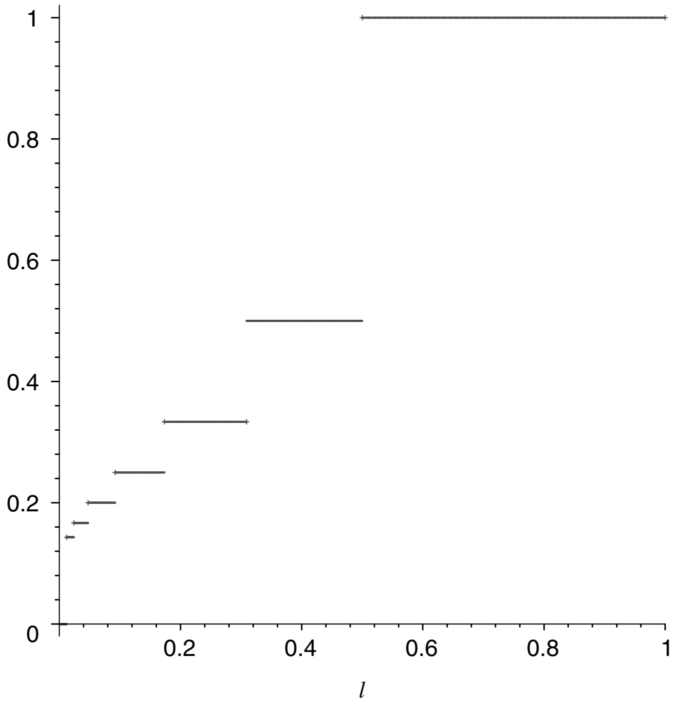

Figure 1. $\alpha(l)$ for the Ulam–von Neumann map.

where $f : SM \to \mathbb{R}$ is defined by $f(x, v) = \chi_A(x)$. That is, we maximize the proportion of time spent in the subset A (of possible *positions*), with no regard to the *direction* (i.e. velocity) of the flow.

An even wider context in which to view the problem of maximum hitting frequencies is the maximization of ergodic averages (1), or in the discrete time case

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

for more general functions f. This ergodic optimization problem has attracted attention from both a specific [B1,HO,J1] and a general [B2,CLT,JMU,YH] point of view. A common feature of these articles is that the reward function f is assumed to be *continuous*. From an applied perspective this is a natural assumption: in physical problems it is usually realistic for rewards to vary continuously. The characteristic functions $f = \chi_A$ treated in this paper are in general not continuous, however; indeed, they represent the most natural class of non-continuous reward functions. Applications are most likely to be found in the social sciences, for example, where rewards (or penalties) are often characterized by an ‘all or nothing’ 0–1 type law.

The organization of this paper is as follows. The definitions and basic theory of maximum hitting frequencies are developed in section 2. The main result here is that if $T : X \to X$ is a continuous map on a compact metric space, then the maximum hitting frequency of a closed set A is an *upper semi-continuous* function of A. Here the topology on the collection of non-empty closed subsets of X is the usual one, as given by the Hausdorff metric. The alternative interpretation of maximum hitting frequency in terms of fastest mean return time is described in section 3. In section 4, we develop some techniques for precisely identifying the (orbits of) maximum hitting frequency for a given set A. These techniques are motivated by the specific examples analysed in sections 5 and 6. The dynamics for these examples are one-dimensional maps (the tent map, Ulam–von Neumann map, continued fraction map and angle-doubling map), while the set $A = A_l$ varies within the one-parameter family of closed intervals defined above. Lastly, in section 6, we briefly discuss the behaviour of the maximum hitting frequency as A varies within the (two-parameter) family of all closed intervals.

2. Maximum hitting frequency

From now on, all our dynamical systems will be discrete, given by iteration of a self-map T of a set X. As we shall see, the most satisfactory development is possible when X is a compact metric space and T is continuous, but for the moment we do not need these hypotheses.

Given any subset A of X, we wish to define the *maximum hitting frequency* associated with A. For a given point $x \in X$, the limit $\lim_{n\to\infty}(1/n)\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\}$ need not exist, so the following two definitions are both equally natural.

Definition 1. *For any subset A of X, define*

$$\gamma(A) = \sup_{x\in X} \limsup_{n\to\infty} \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\} \tag{2}$$

and

$$\beta(A) = \sup_{x\in X_A} \lim_{n\to\infty} \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\}, \tag{3}$$

where X_A denotes the set of $x \in X$ for which the limit $\lim_{n\to\infty}(1/n)\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\}$ does exist. If X_A is empty, then by convention we define $\beta(A) = 0$.

It is not hard to construct examples where the quantities $\beta(A)$ and $\gamma(A)$ do not coincide. We shall be more interested in studying cases where they *do* coincide, however, and to this end we introduce a third notion of maximum hitting frequency, in terms of T-invariant probability measures on X. To deal with such measures we need an appropriate σ-algebra on which to define them. Henceforth, therefore, we assume that X is a topological space and that all measures are Borel (i.e. defined on the Borel σ-algebra of X). Let $\mathcal{M}_T$ denote the set of T-invariant Borel probability measures. In general, $\mathcal{M}_T$ might be empty, though if X is a compact metric space, and T is continuous, then $\mathcal{M}_T$ is non-empty (an observation of Krylov and Bogolioubov, see [Wal, corollary 6.9.1]). If A is a Borel subset of X, then define

$$\alpha(A) = \sup_{\mu\in\mathcal{M}_T} \mu(A), \tag{4}$$

the largest measure assigned to A by a T-invariant probability measure. If $\mathcal{M}_T = \emptyset$, then we set $\alpha(A) = 0$.

Birkhoff's ergodic theorem suggests that $\alpha(A)$ may be related to $\beta(A)$ and $\gamma(A)$. Without any further hypothesis on the subset A, however, the various maximum hitting frequencies $\alpha(A)$, $\beta(A)$, $\gamma(A)$ need not coincide, even when $T : X \to X$ is a continuous map on a compact metric space. For example, if $T(x) = x/2$ on $[0, 1]$ and $A = (0, 1]$, then $\beta(A) = \gamma(A) = 1$ since there are orbits which stay in A for all time, whereas $\alpha(A) = 0$ because the only T-invariant probability measure is the one supported on the fixed point 0. To avoid pathologies of this kind, associated with the presence of recurrent dynamics on the boundary of the subset A, we shall henceforth require that A be a *closed* set. For a continuous self-map of a compact metric space, we then have the following proposition.

Proposition 1. *Let $T : X \to X$ be a continuous map on a compact metric space and A a closed subset of X. Then*

$$\alpha(A) = \beta(A) = \gamma(A) \tag{5}$$

and there is at least one T-invariant probability measure μ for which $\mu(A) = \alpha(A)$.

Proof. The inequality $\beta(A) \leqslant \gamma(A)$ is immediate from the definitions (2) and (3).

We next show that $\gamma(A) \leqslant \alpha(A)$. For any $x \in X$, write

$$\gamma_x(A) = \limsup_{n\to\infty} \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\}$$

and let n_j be an increasing sequence such that

$$\gamma_x(A) = \lim_{j\to\infty} \frac{1}{n_j}\#\{0 \leqslant i \leqslant n_j - 1 : T^i(x) \in A\}.$$

If we define the Borel probability measure μ_j by $\mu_j = 1/n_j \sum_{i=0}^{n_j-1} \delta_{T^i x}$, then $\gamma_x(A) = \lim_{j\to\infty} \mu_j(A)$. The compactness of X means that the set $\mathcal{M}$ of Borel probability measures on X is compact with respect to the weak* topology (cf [Wal, theorem 6.5]). So there exists a measure $\mu \in \mathcal{M}$, and an increasing subsequence j_k, such that $\mu_{j_k} \to \mu$ in the weak* topology. Since A is closed it follows (see [Bil, theorem 2.1]) that $\lim_{k\to\infty} \mu_{j_k}(A) \leqslant \mu(A)$. Therefore, $\gamma_x(A) \leqslant \mu(A)$. From the definition of μ_j it is easily shown that μ is T-invariant, so $\gamma_x(A) \leqslant \sup_{m\in\mathcal{M}_T} m(A) = \alpha(A)$ for every $x \in X$ and hence $\gamma(A) \leqslant \alpha(A)$.

To see that $\alpha(A) \leqslant \beta(A)$, suppose on the contrary that there exists an invariant measure $\mu \in \mathcal{M}_T$ for which $\mu(A) > \beta(A)$. The ergodic decomposition theorem [Wal, pp 34, 153, remark (2)] means we may assume μ to be ergodic, and Birkhoff's ergodic theorem then guarantees the existence of an $x \in X$ for which

$$\mu(A) = \lim_{n\to\infty} \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : T^i(x) \in A\} \leqslant \beta(A),$$

a contradiction.

Since A is closed, the functional $\mu \mapsto \mu(A)$ on $\mathcal{M}_T$ is upper semi-continuous with respect to the weak* topology [Bil, theorem 2.1], and since $\mathcal{M}_T$ is compact for this topology the supremum $\alpha(A) = \sup_{m\in\mathcal{M}_T} m(A)$ is attained at some $\mu \in \mathcal{M}_T$. ☐

In view of proposition 1, we make the following definition.

Definition 2. *Let $T : X \to X$ be a continuous map on a compact metric space and A a closed subset of X. The maximum hitting frequency for the set A is the value $\alpha(A)$ defined, as in (4), by*

$$\alpha(A) = \sup_{\mu\in\mathcal{M}_T} \mu(A)$$

or equivalently by the right-hand side of either (2) or (3). Let $\mathcal{M}_{\max}(A)$ denote the non-empty set of T-invariant measures μ for which $\mu(A) = \alpha(A)$. We say that any measure in $\mathcal{M}_{\max}(A)$ has maximum hitting frequency for A, or that it hits A with maximum frequency.

A natural problem is to determine the continuity properties of the map $A \mapsto \alpha(A)$. For this it will be useful to make sense of $\alpha(f)$ when $f : X \to \mathbb{R}$ is a *function*. The natural definition is $\alpha(f) = \sup_{\mu\in\mathcal{M}_T} \int f \, d\mu$, provided f is Borel measurable and bounded, say. This is an extension of definition 2, in the sense that if χ_A is the characteristic function of A, then

$\alpha(\chi_A) = \alpha(A)$. Irrespective of whether its argument is a set or a function, $\alpha(\cdot)$ clearly enjoys the following monotonicity property.

Lemma 1. *Let $T : X \to X$ be a continuous map on a compact metric space. If $f, g : X \to \mathbb{R}$ are bounded and Borel measurable, and $f(x) \leqslant g(x)$ for all $x \in X$, then $\alpha(f) \leqslant \alpha(g)$.*

In particular if A, B are closed sets with $A \subset B$, then $\alpha(A) \leqslant \alpha(B)$.

Now let $C(X)$ denote the space of continuous real-valued functions on X, equipped with the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. As a functional on $C(X)$, α is rather well-behaved.

Lemma 2. *Let $T : X \to X$ be a continuous map on a compact metric space. The functional $\alpha : C(X) \to \mathbb{R}$, defined by*

$$\alpha(f) = \sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu$$

is continuous.

Indeed, it is easily seen that α is 1-Lipschitz on $C(X)$: if $\mu_i \in \mathcal{M}_T$ is such that $\alpha(f_i) = \int f_i \,\mathrm{d}\mu_i$, for $i = 1, 2$, then

$$\alpha(f_1) - \alpha(f_2) = \int f_1 \,\mathrm{d}\mu_1 - \int f_2 \,\mathrm{d}\mu_2 \leqslant \int (f_1 - f_2) \,\mathrm{d}\mu_1 \leqslant \|f_1 - f_2\|_\infty.$$

Lemmas 1 and 2 will be useful in studying the regularity of α as a function defined on the collection $K(X)$ of closed non-empty subsets of X. This is itself a compact metric space when equipped with the *Hausdorff metric* D, defined by

$$D(A, B) = \max\left\{\max_{x\in A} d(x, B),\ \max_{x\in B} d(x, A)\right\}.$$

Here $d(x, A) := \min_{a\in A} d(x, a)$, where d is the metric on X.

In general, α will not be a *continuous* function on $K(X)$. We have already seen one example of this in the introduction, and others will be described in sections 5 and 6. An even simpler example is to take a contracting map T, for example $T(x) = x/2$ on $[0, 1]$; here it is easily verified that α is discontinuous at the singleton set containing the unique fixed point of T.

It turns out, however, that $\alpha : K(X) \to \mathbb{R}$ is always *upper semi-continuous*. In other words, if $A_n \to A$ in $K(X)$ then $\limsup_n \alpha(A_n) \leqslant \alpha(A)$. A proof of this result (theorem 1) will occupy the remainder of this section. The first step is the following lemma concerning the behaviour of α under *monotone* convergence.

Lemma 3. *Let $T : X \to X$ be a continuous map on a compact metric space. Let $A \subset X$ be closed, and for $\varepsilon > 0$ define the closed set A^ε by*

$$A^\varepsilon = \{x \in X : d(x, A) \leqslant \varepsilon\}.$$

Then

$$\alpha(A) = \lim_{\varepsilon\to 0} \alpha(A^\varepsilon). \tag{6}$$

Proof. The sets A^ε are decreasing as $\varepsilon \to 0$, so $\alpha(A^\varepsilon)$ is non-increasing, by lemma 1. Since $\alpha(A^\varepsilon)$ is bounded below by 0 it follows that $\lim_{\varepsilon\to 0} \alpha(A^\varepsilon)$ exists.

Now, $A \subset A^\varepsilon$ for all $\varepsilon > 0$, so $\alpha(A) \leqslant \alpha(A^\varepsilon)$ by lemma 1, and therefore

$$\alpha(A) \leqslant \lim_{\varepsilon\to 0} \alpha(A^\varepsilon). \tag{7}$$

It remains to show that

$$\alpha(A) \geqslant \lim_{\varepsilon\to 0} \alpha(A^\varepsilon). \tag{8}$$

Let $\mu_\varepsilon \in \mathcal{M}_{\max}(A^\varepsilon)$, and let μ be any weak* accumulation point of the family μ_ε as $\varepsilon \to 0$. Clearly, $\mu \in \mathcal{M}_T$. Let $\delta > 0$ be arbitrary. If $0 < \varepsilon < \delta$ then $A^\varepsilon \subset A^\delta$, so

$$\mu_\varepsilon(A^\varepsilon) \leqslant \mu_\varepsilon(A^\delta).$$

Taking limit suprema gives

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(A^\varepsilon) \leqslant \limsup_{\varepsilon\to 0} \mu_\varepsilon(A^\delta). \tag{9}$$

By [Bil, theorem 2.1] we know that

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(A^\delta) \leqslant \mu(A^\delta), \tag{10}$$

so combining (9) and (10) yields

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(A^\varepsilon) \leqslant \mu(A^\delta) \qquad \text{for any } \delta > 0. \tag{11}$$

But μ is a regular measure [Bil, theorem 1.1], so $\mu(A^\delta) \to \mu(A)$ as $\delta \to 0$. Moreover $\mu \in \mathcal{M}_T$, so $\mu(A) \leqslant \alpha(A)$. Together with (11) these observations give

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(A^\varepsilon) \leqslant \lim_{\delta\to 0} \mu(A^\delta) = \mu(A) \leqslant \alpha(A), \tag{12}$$

which is the required inequality (8). □

Remark 1. Combining the inequalities (7) and (12), we see that $\mu(A) = \alpha(A)$. In other words, if μ is any weak* limit point of measures μ_ε which hit A^ε with maximum frequency, then μ hits A with maximum frequency.

The following theorem is the main general result concerning the regularity of the maximum hitting frequency $\alpha : K(X) \to \mathbb{R}$.

Theorem 1. *Let $T : X \to X$ be a continuous map on a compact metric space. The maximum hitting frequency $\alpha : K(X) \to [0, 1]$ is upper semi-continuous.*

Proof. If $A_n \to A$ in $K(X)$, we must show that

$$\alpha(A) \geqslant \limsup_{n\to\infty} \alpha(A_n). \tag{13}$$

For this it suffices to prove that

$$\alpha(A^\varepsilon) \geqslant \limsup_{n\to\infty} \alpha(A_n) \tag{14}$$

for all $\varepsilon > 0$, since $\alpha(A) = \lim_{\varepsilon\to 0} \alpha(A^\varepsilon)$ by lemma 3.

Define, for any $\varepsilon > 0$, the continuous functions f^ε and f_n^ε by

$$f^\varepsilon(x) = \left(1 - \frac{d(x, A)}{\varepsilon}\right)^+, \qquad f_n^\varepsilon(x) = \left(1 - \frac{d(x, A_n)}{\varepsilon}\right)^+.$$

These are approximations to the characteristic functions of A and A_n in the sense that

$$\chi_{A^\varepsilon} \geqslant f^\varepsilon \geqslant \chi_A \tag{15}$$

and

$$\chi_{A_n^\varepsilon} \geqslant f_n^\varepsilon \geqslant \chi_{A_n}. \tag{16}$$

We shall prove shortly that $f_n^\varepsilon \to f^\varepsilon$ in $C(X)$ as $n \to \infty$, from which it follows that $\alpha(f^\varepsilon) = \lim_{n\to\infty} \alpha(f_n^\varepsilon)$, by lemma 2. Combining this with the left-hand side of (15), the right-hand side of (16) and lemma 1, we deduce that

$$\begin{aligned}\alpha(A^\varepsilon) = \alpha(\chi_{A^\varepsilon}) \geqslant \alpha(f^\varepsilon) &= \lim_{n\to\infty} \alpha(f_n^\varepsilon) \\ &\geqslant \limsup_{n\to\infty} \alpha(\chi_{A_n}) \\ &= \limsup_{n\to\infty} \alpha(A_n),\end{aligned}$$

so that (14) is proved.

It remains to justify that $f_n^\varepsilon \to f^\varepsilon$ in $C(X)$ as $n \to \infty$. In fact, we claim that

$$\|f_n^\varepsilon - f^\varepsilon\|_\infty \leqslant \frac{D(A, A_n)}{\varepsilon}. \tag{17}$$

To check this we shall consider the various possible locations of a point x in X and prove that in each case

$$|(f_n^\varepsilon - f^\varepsilon)(x)| \leqslant \frac{D(A, A_n)}{\varepsilon}. \tag{18}$$

First, if $x \in X$ is such that $d(x, A) > \varepsilon$ and $d(x, A_n) > \varepsilon$ then both f^ε and f_n^ε are identically zero, so (18) certainly holds. Second, if both

$$d(x, A_n) \leqslant \varepsilon \tag{19}$$

and

$$d(x, A) \leqslant \varepsilon \tag{20}$$

are satisfied then

$$|f_n^\varepsilon(x) - f^\varepsilon(x)| = \frac{|d(x, A) - d(x, A_n)|}{\varepsilon} \leqslant \frac{D(A, A_n)}{\varepsilon}.$$

Lastly, if (19) fails but (20) holds then

$$|f_n^\varepsilon(x) - f^\varepsilon(x)| = 1 - \frac{d(x, A)}{\varepsilon} \leqslant 1 - \frac{\varepsilon - D(A, A_n)}{\varepsilon} = \frac{D(A, A_n)}{\varepsilon}$$

and similarly if (19) holds but (20) fails then

$$|f_n^\varepsilon(x) - f^\varepsilon(x)| = 1 - \frac{d(x, A_n)}{\varepsilon} \leqslant 1 - \frac{\varepsilon - D(A, A_n)}{\varepsilon} = \frac{D(A, A_n)}{\varepsilon}.$$

So (18) holds for all $x \in X$, and therefore (17) is true. □

3. Fastest mean return time

If an orbit hits the set A with maximum frequency, it is intuitively obvious that its *average return time* to A will be smaller than for other orbits. Here we clarify this assertion, though not in full detail; the interested reader should be able to fill in any gaps using the ideas from section 2. We shall assume throughout that A is a closed subset of a compact metric space and $T : X \to X$ is continuous.

If the orbit of x hits A infinitely often, then let $N_x(0) < N_x(1) < \cdots$ be an exhaustion of those instances for which $T^{N_x(k)}(x) \in A$. This is the sequence of *return times* to A of the orbit generated by x. The corresponding *mean return time* to A is defined as

$$r_x(A) = \lim_{k\to\infty} \frac{N_x(k)}{k}$$

provided this limit exists. The *fastest mean return time* to A, denoted $r(A)$, is defined as the infimum of $r_x(A)$ over all points x whose mean return time is well-defined. Alternatively, and equivalently, we could define $r(A) = \inf_{x\in X} \liminf_{k\to\infty}(N_x(k)/k)$.

The counting function $C_x(j) = \#\{0 \leqslant i \leqslant j-1 : T^i(x) \in A\}$ is approximately inverse to N_x, in the sense that $N_x(C_x(k)) \leqslant k < N_x(C_x(k)+1)$. This observation can be used to prove the relation

$$r(A) = \frac{1}{\alpha(A)} \tag{21}$$

between fastest mean return time and optimal hitting frequency.

A definition of fastest mean return time in terms of invariant measures is also possible. If $\mu \in \mathcal{M}_T$ is such that $\mu(A) > 0$, then Poincaré's recurrence theorem guarantees that μ-almost every point in A returns to A under iteration by T. So the *(first) return time function*

$$n_A(x) = \inf\{n > 0 : T^n(x) \in A\}$$

is defined, and finite, for μ-almost every $x \in A$. The *mean return time* to A for the measure μ is defined as

$$r_\mu(A) = \frac{1}{\mu(A)} \int_A n_A \,\mathrm{d}\mu$$

and it can be shown [Wr] that

$$r_\mu(A) = \frac{\mu(\bigcup_{n\geqslant 1} T^{-n}A)}{\mu(A)}.$$

In particular, if μ is ergodic then $\mu(\bigcup_{n\geqslant 1} T^{-n}A) = 1$, so that $r_\mu(A) = 1/\mu(A)$, a formula first obtained by Kac [Kac].

The *fastest mean return time to A* is defined as

$$r(A) = \inf_{\mu\in\mathcal{M}_T} r_\mu(A).$$

Any invariant measure attaining this infimum will be said to have the *fastest mean return time to A*. The ergodic decomposition theorem implies that the infimum is attained at some ergodic measure, so in fact

$$r(A) = \inf_{\mu\in\mathcal{E}_T} \frac{1}{\mu(A)},$$

where $\mathcal{E}_T$ denotes the set of ergodic T-invariant probability measures. Therefore

$$r(A) = \frac{1}{\sup_{\mu\in\mathcal{E}_T} \mu(A)} = \frac{1}{\sup_{\mu\in\mathcal{M}_T} \mu(A)} = \frac{1}{\alpha(A)},$$

which is equation (21) again.

So far we have assumed that $\alpha(A) > 0$. If on the other hand $\alpha(A) = 0$, then no orbit, or invariant measure, hits A with positive frequency. So the mean return time to A is always infinite, and we therefore set $r(A) = \infty$. That is, the equality (21) holds even when $\alpha(A) = 0$.

In summary we have the following proposition.

Proposition 2. *Let $T : X \to X$ be a continuous map on a compact metric space and A a closed subset of X.*

The fastest mean return time to A is the reciprocal $1/\alpha(A)$ of the maximum hitting frequency $\alpha(A)$.

In particular, a measure $\mu \in \mathcal{M}_T$ has the fastest mean return time to A if and only if $\mu \in \mathcal{M}_{\max}(A)$.

4. Identifying maximum hitting frequencies

The following lemma will be used to analyse the maximum hitting frequency for certain subsets A. Recall that the *(topological) support* of a probability measure μ, denoted $\mathrm{supp}(\mu)$, is the smallest closed subset B such that $\mu(B) = 1$. A subset $Z \subset X$ will be called T*-invariant* if $T(Z) = Z$. If $\mu \in \mathcal{M}_T$, then $\mathrm{supp}(\mu)$ is T-invariant.

Lemma 4. *Let $T : X \to X$ be a continuous map on a compact metric space and A a closed subset of X. Suppose there exists a bounded Borel measurable function f_A such that $\int f_A \, \mathrm{d}\mu = \mu(A)$ for all $\mu \in \mathcal{M}_T$, and such that $f_A^{-1}(\sup f_A)$ contains a non-empty closed T-invariant set.*

Then

$$\alpha(A) = \sup f_A \tag{22}$$

and $\mathcal{M}_{\max}(A)$ consists precisely of those measures $\mu \in \mathcal{M}_T$ whose topological support is contained in the set $f_A^{-1}(\sup f_A)$.

Proof. Since $\mu(A) = \int f_A \, \mathrm{d}\mu$ for all $\mu \in \mathcal{M}_T$,

$$\alpha(A) = \sup_{\mu \in \mathcal{M}_T} \mu(A) = \sup_{\mu \in \mathcal{M}_T} \int f_A \, \mathrm{d}\mu.$$

In particular,

$$\alpha(A) \leqslant \sup f_A \tag{23}$$

and $\mu \in \mathcal{M}_{\max}(A)$ if and only if

$$\int f_A \, \mathrm{d}\mu = \alpha(A).$$

Let Y denote the largest closed T-invariant subset of $f_A^{-1}(\sup f_A)$. By assumption, Y is non-empty. Since $T|_Y : Y \to Y$ is a continuous self-map of a compact metric space, it has at least one $T|_Y$-invariant probability measure, by the Krylov–Bogolioubov theorem [Wal, corollary 6.9.1]. Any such $T|_Y$-invariant probability measure μ can be regarded as a T-invariant measure on X with $\mathrm{supp}(\mu) \subset Y$. In particular, $\mathrm{supp}(\mu) \subset f_A^{-1}(\sup f_A)$, so $\int f_A \, \mathrm{d}\mu = \sup f_A$, and by virtue of (23) we deduce first the required equality (22) and second that $\mu \in \mathcal{M}_{\max}(A)$.

So we have shown that if the support of $\mu \in \mathcal{M}_T$ lies in Y, then $\mu \in \mathcal{M}_{\max}(A)$. Now the support of any T-invariant measure is a closed T-invariant set, so if the support of $\mu \in \mathcal{M}_T$ lies in $f_A^{-1}(\sup f_A)$, then in fact it lies in Y, and hence $\mu \in \mathcal{M}_{\max}(A)$. Conversely, suppose that $\mu \in \mathcal{M}_T$ is such that $\mathrm{supp}(\mu) \not\subset f_A^{-1}(\sup f_A)$. Then $\mu(\{x \in X : f_A(x) < \alpha(A)\}) > 0$, so $\int f_A \, \mathrm{d}\mu < \alpha(A)$, and therefore $\mu \notin \mathcal{M}_{\max}(A)$. ☐

Remark 2. The idea of finding a function f_A as in lemma 4 can be traced back to an unpublished paper by Conze and Guivarc'h [CG], where it is shown that if $T : X \to X$ is a subshift of finite type, and $f : X \to \mathbb{R}$ is Hölder continuous, then there exists a (Hölder) continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}^{-1}(\max \tilde{f})$ contains a closed T-invariant set and $\int \tilde{f} \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$ for all $\mu \in \mathcal{M}_T$. Analogous results have subsequently been discovered in other contexts where the dynamics has some hyperbolicity and the continuous function f is sufficiently regular (see, e.g., [B1, B2, CLT, J2, JMU]).

The following application of lemma 4 will be the key tool in the analysis of the specific examples of sections 5 and 6.

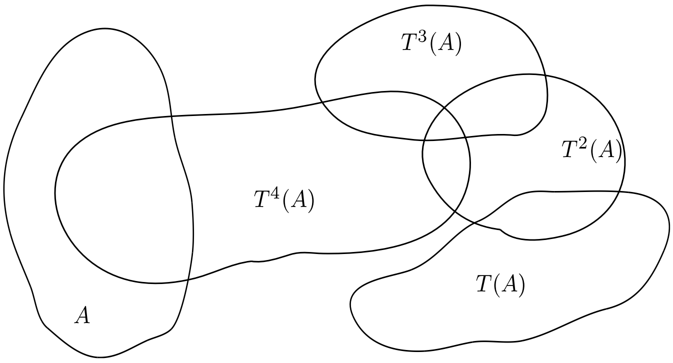

Figure 2. Case $n = 4$ in proposition 3.

Proposition 3. *Let $T : X \to X$ be a continuous map on a compact metric space and A a closed subset of X. Suppose that for some $n \geqslant 2$, the images $T(A), \ldots, T^{n-1}(A)$ are all disjoint from A but the closed T^n-invariant set $A^{(n)} = \bigcap_{i=0}^{\infty} T^{-in}(A)$ is non-empty. Then*

- *(i)* $\alpha(A) = 1/n$,
- *(ii)* *the T-invariant probability measures with maximum hitting frequency for A are precisely those whose topological support is contained in the closed T-invariant set*

$$A^{(n)} \cup T(A^{(n)}) \cup \cdots \cup T^{n-1}(A^{(n)}).$$

Proof. Define

$$f_A = \frac{1}{n}\sum_{i=0}^{n-1} \chi_{T^{-i}A}$$

and note that for all $\mu \in \mathcal{M}_T$,

$$\int f_A \,\mathrm{d}\mu = \frac{1}{n}\sum_{i=0}^{n-1}\int \chi_{T^{-i}A}\,\mathrm{d}\mu = \frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}A) = \frac{1}{n}\sum_{i=0}^{n-1}\mu(A) = \mu(A).$$

Now the inverse images $A, T^{-1}(A), \ldots, T^{-(n-1)}(A)$ are pairwise disjoint, since if $T^{-i}(A) \cap T^{-j}(A)$ were non-empty for some $0 \leqslant i < j \leqslant n-1$ then $T^{j-i}(A)$ would intersect A contradicting our hypothesis. Therefore

$$f_A = \frac{1}{n}\sum_{i=0}^{n-1}\chi_{T^{-i}A} = \frac{1}{n}\chi_{A\cup T^{-1}(A)\cup\cdots\cup T^{-(n-1)}A}.$$

This function attains its maximum value $1/n$ on, and only on, the set $M_n = A \cup T^{-1}(A) \cup \cdots \cup T^{-(n-1)}A$. We would like to apply lemma 4, so we must verify that M_n contains a non-empty closed T-invariant set. That is, we must check that $Y = \bigcap_{j=0}^{\infty} T^{-j}(M_n)$ is non-empty.

Since $A, T^{-1}(A), \ldots, T^{-(n-1)}(A)$ are pairwise disjoint, if $J = kn + r$ then

$$\bigcap_{j=0}^{J} T^{-j}(M_n) = \left(\bigcup_{l=0}^{r-1}\bigcap_{i=0}^{k+1} T^{-in+l}(A)\right) \cup \left(\bigcup_{l=r}^{n-1}\bigcap_{i=0}^{k} T^{-in+l}(A)\right).$$

Therefore

$$Y = \bigcap_{j=0}^{\infty} T^{-j}(M_n) = \bigcup_{l=0}^{n-1}\bigcap_{i=0}^{\infty} T^{-in+l}(A) = \bigcup_{l=0}^{n-1} T^{l}(A^{(n)}),$$

so Y is non-empty, since by hypothesis $A^{(n)}$ is non-empty.

The hypotheses of lemma 4 are therefore satisfied, so we deduce that

$$\alpha(A) = \sup f_A = \sup \frac{1}{n}\chi_{A\cup T^{-1}(A)\cup\cdots\cup T^{-(n-1)}A} = \frac{1}{n}$$

and that the T-invariant measures with maximum hitting frequency are precisely those whose topological support is contained in $M_n = A \cup T^{-1}(A) \cup \cdots \cup T^{-(n-1)}A$. Since these supports must be closed and T-invariant, they are in fact contained in $Y = \bigcap_{j=0}^{\infty} T^{-j}(M_n) = \bigcup_{l=0}^{n-1} T^l(A^{(n)})$, as required. $\square$

5. Interval maps

In this section, we shall take T to be various well-known maps of the interval and consider maximum hitting frequencies for certain closed sub-intervals A. In fact, we shall be interested in parametrized families of sub-intervals, our main example being the centrally symmetric family $A_l = [(1-l)/2, (1+l)/2]$. The maximum hitting frequency $\alpha(A_l)$ will simply be denoted $\alpha(l)$.

5.1. The tent map

Our first dynamical system is the (full) *tent map* $T : [0, 1] \to [0, 1]$, defined by

$$T(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2-2x & \text{for } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

The maximum hitting frequencies $\alpha(l) = \alpha(A_l)$ for the centrally symmetric family $A_l = [(1-l)/2, (1+l)/2]$ are described by the following result (see also figure 3).

Theorem 2. *If T is the tent map then*

$$\alpha(l) = \frac{1}{n} \qquad \textit{for } l \in [l_n, l_{n-1}),$$

where $l_0 = 1$ and $l_n = 1/(2^n + 1)$ for $n \geqslant 1$.

The period-n orbit

$$\left\{ \frac{2}{2^n+1}, \ldots, \frac{2^{n-1}}{2^n+1}, \frac{2^n}{2^n+1} \right\}$$

hits $A_l = [(1-l)/2, (1+l)/2]$ with maximum frequency, for all $l \in [l_n, l_{n-1})$.

Proof. Consider first the case $n = 1$. If $1/3 = l_1 \leqslant l < l_0 = 1$ then the fixed point $2/3$ lies in A_l, so $\alpha(l) = 1$.

Now suppose $n \geqslant 2$. If $l < l_{n-1} = 1/(2^{n-1}+1)$, then $TA_l = [1-l, 1]$, and $T^i A_l = [0, 2^{i-1}l]$ for $2 \leqslant i \leqslant n-1$. These first $n-1$ images of A_l are disjoint from A_l because

$$1 - l > \frac{1}{2} + \frac{l}{2}$$

and

$$2^{n-2}l < \frac{2^{n-2}}{2^{n-1}+1} = \frac{1}{2} - \frac{1}{2(2^{n-1}+1)} < \frac{1}{2} - \frac{l}{2}.$$

On the other hand, the point $2^{n-1}/(2^n+1)$ is of period n and is contained in A_l for all $l \geqslant l_n = 1/(2^n+1)$, so $A_l^{(n)} = \bigcap_{i=0}^{\infty} T^{-in}(A_l)$ is non-empty. It follows from proposition 3 that $\alpha(l) = \alpha(A_l) = 1/n$ for $1/(2^n+1) = l_n \leqslant l < l_{n-1} = 1/(2^{n-1}+1)$, as required.

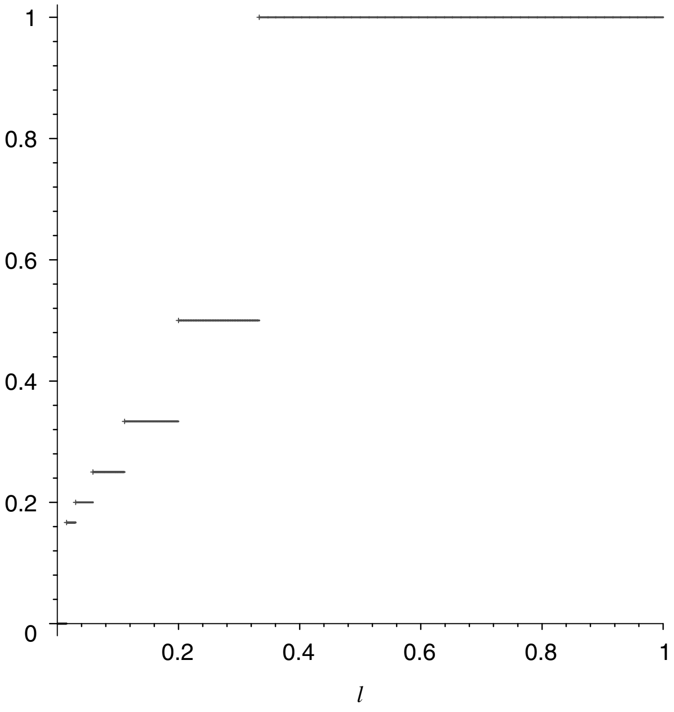

Figure 3. $\alpha(l)$ for the tent map.

Also by proposition 3, the T-invariant probability measures which hit A_l most frequently are precisely those whose topological support is contained in

$$A_l^{(n)} \cup T(A_l^{(n)}) \cup \cdots \cup T^{n-1}(A_l^{(n)}).$$

As we have just seen, the invariant measure supported on

$$\left\{ \frac{2}{2^n+1}, \ldots, \frac{2^{n-1}}{2^n+1}, \frac{2^n}{2^n+1} \right\}$$

is one such measure. ☐

5.2. *The Ulam–von Neumann map*

Now let T denote the Ulam–von Neumann map $T(x) = 4x(1-x)$. This is well known to be topologically conjugate to the tent map. Explicitly, if $h : [0,1] \to [0,1]$ is the homeomorphism defined by $h(x) = \frac{1}{2}(1-\cos \pi x)$, then $h \circ S = T \circ h$, where now S denotes the tent map. Note that h maps the length-l centrally symmetric interval $A_l = [\frac{1}{2}(1-l), \frac{1}{2}(1+l)]$ to the centrally symmetric interval $[\frac{1}{2}(1-\sin(\pi l/2)), \frac{1}{2}(1+\sin(\pi l/2))]$ of length $\sin(\pi l/2)$. It follows from theorem 2 that the maximum hitting frequencies are given as follows (see also figure 1):

Theorem 3. *If T is the Ulam–von Neumann map $T(x) = 4x(1-x)$, then*

$$\alpha(l) = \frac{1}{n} \qquad \text{for } l \in [l_n, l_{n-1}),$$

where

$$l_0 = 1 \quad \text{and} \quad l_n = \sin\left(\frac{\pi}{2(2^n+1)}\right) \qquad \text{for } n \geqslant 1.$$

5.3. The Gauss map

Now let T be Gauss's continued fraction map, defined on the set of irrationals in the unit interval by the formula $T(x) = 1/x \pmod 1$. We might also extend T to the whole of $[0, 1]$ by this formula, and by setting $T(0) = 0$, so that every rational is homoclinic to this new fixed point. Whether or not we make this extension does not affect the maximum hitting frequencies $\alpha(l)$ for the centrally symmetric intervals A_l, which are described as follows.

Theorem 4. *If* $T(x) = 1/x \pmod 1$ *is Gauss's continued fraction map then*

$$\alpha(l) = \begin{cases} 1 & \text{for } l \geqslant \frac{3}{2} - \sqrt{2}, \\ \frac{1}{2} & \text{for } 0 < l < \frac{3}{2} - \sqrt{2}. \end{cases}$$

Proof. If $l \geqslant \frac{3}{2} - \sqrt{2}$, then the fixed point

$$\sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

lies in the interval A_l, so $\alpha(l) = 1$.

If $0 < l < \frac{3}{2} - \sqrt{2}$, then $T(A_l)$ is disjoint from A_l, as is easily verified. On the other hand, there exist period-2 points arbitrarily close to the point $1/2$. Specifically, the quadratic irrationals

$$\frac{1}{2}\sqrt{n^2 + 2n} - \frac{n}{2} = \cfrac{1}{2 + \cfrac{1}{n + \cfrac{1}{2 + \cfrac{1}{n + \cdots}}}}$$

converge to $1/2$ as $n \to \infty$, so A_l will always contain such points. Therefore $A_l^{(2)} = \bigcap_{i=0}^{\infty} T^{-2i} A_l$ is non-empty, and by proposition 3(i) we deduce that $\alpha(l) = 1/2$. ☐

6. The angle-doubling map

Let T denote the degree-2 circle map $T(x) = 2x \pmod 1$, and again let $A_l = [(1 - l)/2, (1 + l)/2]$ and $\alpha(l) = \alpha(A_l)$. It turns out that the function $\alpha(l)$ is *identical* to its tent map analogue, though of course the orbits with maximum hitting frequency are different.

Theorem 5. *If* T *is the angle-doubling map* $T(x) = 2x \pmod 1$ *then*

$$\alpha(l) = \frac{1}{n} \qquad \text{for } l \in [l_n, l_{n-1}),$$

where $l_0 = 1$ *and* $l_n = 1/(2^n + 1)$ *for* $n \geqslant 1$.

The period-$2n$ *orbit generated by the point* $1/(2^n + 1)$ *hits* A_l *with maximum frequency for all* $1/(2^n + 1) = l_n \leqslant l < l_{n-1} = 1/(2^{n-1} + 1)$.

If $1/(2^n + 1) = l_n \leqslant l \leqslant 1/2^n$ *then this period-*$2n$ *orbit is the unique measure with maximum hitting frequency.*

Proof. If $1/3 = l_1 \leqslant l < l_0 = 1$, then the period-2 orbit $\{1/3, 2/3\}$ is contained in A_l, so $\alpha(l) = 1$.

Now suppose that $n \geqslant 2$. If $l < l_{n-1} = 1/(2^{n-1}+1)$ then $T^i A_l = [0, 2^{i-1}l] \cup [1-2^{i-1}l, 1]$ (i.e. the interval stretching counter-clockwise on the circle from $1 - 2^{i-1}l$ to $2^{i-1}l$) for all

$1 \leqslant i \leqslant n-1$. The images $T(A_l), \ldots, T^{n-1}(A_l)$ are all disjoint from A_l, since

$$2^{n-2} l < \frac{2^{n-2}}{2^{n-1}+1} = \frac{1}{2} - \frac{1}{2(2^{n-1}+1)} < \frac{1}{2} - \frac{l}{2}.$$

The orbit generated by the point $x_n = 1/(2^n+1)$ is of period $2n$, and in particular contains the points $2^{n-1}/(2^n+1) = T^{n-1}(x_n)$ and $(2^{n-1}+1)/(2^n+1) = T^{2n-1}(x_n)$. Both these points are distance $1/(2(2^n+1))$ from $1/2$, so are contained in $A_l = [(1-l)/2, (1+l)/2]$ for all $l \geqslant l_n = 1/(2^n+1)$. So $A_l^{(n)} = \bigcap_{i=0}^{\infty} T^{-in}(A_l)$ is non-empty: it contains the points $2^{n-1}/(2^n+1)$ and $(2^{n-1}+1)/(2^n+1)$. It follows from proposition 3(i) that $\alpha(l) = \alpha(A_l) = 1/n$ for $1/(2^n+1) = l_n \leqslant l < l_{n-1} = 1/(2^{n-1}+1)$, as required.

By proposition 3(ii), the T-invariant probability measures which hit A_l most frequently are precisely those whose topological support is contained in

$$A_l^{(n)} \cup T(A_l^{(n)}) \cup \cdots \cup T^{n-1}(A_l^{(n)}).$$

In particular, the invariant measure supported on the period-$2n$ orbit generated by x_n has maximum hitting frequency.

It remains to show that if $1/(2^n+1) \leqslant l \leqslant 1/2^n$ then the measure generated by x_n is the *unique* one with maximum hitting frequency. For this we need the fact that every length-$1/2^n$ closed sub-interval of the circle contains the support of one and only one T^n-invariant probability measure. This fact is contained in [HJ] and can be proved using arguments analogous to those in [BS], where the case $n = 1$ is treated. In the particular case where the length-$1/2^n$ sub-interval is $[(1-2^{-n})/2, (1+2^{-n})/2]$, the support in question is the set $\{2^{n-1}/(2^n+1), (2^{n-1}+1)/(2^n+1)\}$, which is a period-2 orbit for T^n. So $A_l^{(n)} = \{2^{n-1}/(2^n+1), (2^{n-1}+1)/(2^n+1)\}$ for all $1/(2^n+1) \leqslant l \leqslant 1/2^n$. The set $A_l^{(n)} \cup T(A_l^{(n)}) \cup \cdots \cup T^{n-1}(A_l^{(n)})$ is therefore *precisely* the period-$2n$ T-orbit generated by the point $x_n = 1/(2^n+1)$, and by proposition 3(ii) this is the unique measure with maximum hitting frequency. □

Remark 3. (a) In the case that $l \in [1/(2^n+1), 1/2^n]$, theorem 5 says that there is a unique measure which hits A_l with maximum frequency. If l is not in one of these intervals, then there may be *many* measures of maximum hitting frequency. In general, these measures are not supported on a periodic orbit; indeed they may have positive entropy. For example, this is clearly the case if l is sufficiently close to 1. It is also the case whenever l is in the range $1/2^{n+1} < l < 1/(2^n+1)$ and sufficiently close to $1/(2^n+1)$. This can be shown by analysing those invariant measures for the map $T^n(x) = 2^n x \pmod 1$ whose support is contained in A_l. If $A_l^{(n)}$ denotes the largest closed T^n-invariant subset of A_l and $B_l := A_l^{(n)} \cup T(A_l^{(n)}) \cup \cdots \cup T^{n-1}(A_l^{(n)})$, then the ($T$-invariant) measures of maximum hitting frequency are those whose support lies in B_l. A closer analysis of the sets B_l or indeed the entropy function

$$H : l \mapsto h_{\mathrm{top}}(T|B_l) = \sup\{h(\mu) : \mu \in \mathcal{M}_{\max}(A_l)\}$$

is possible using the techniques of Urbański [Urb1, Urb2] applied to the map T^n. Both B_l and $H(l)$ are clearly piecewise monotone on each interval of constancy of α, though [Urb1, Urb2] implies they are locally constant on certain sub-intervals. The entropy function H is continuous on each interval of constancy of α, albeit singular (a 'devil's staircase'), by [Urb2, theorem 4].

(b) The family of centrally symmetric closed intervals $A_l = [(1-l)/2, (1+l)/2]$ is just a one-parameter section of the two-parameter family $A_{c,l} = [c - l/2, c + l/2]$ of all closed intervals. The sections orthogonal to $l \mapsto \alpha(A_l)$ correspond to translating an interval of fixed length around the circle. Such one-parameter families $c \mapsto \alpha(A_{c,l})$ are easy to describe if the interval has length $l \geqslant 1/2$: in that case $\alpha(A_{c,l}) = 1$ for all c. This is because every closed semi-circle S contains the support of some T-invariant measure[2], since S is compact

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[2] In fact, more is known, as mentioned in the proof of theorem 5: every closed semi-circle contains the support of a *unique* invariant measure, cf [BS].

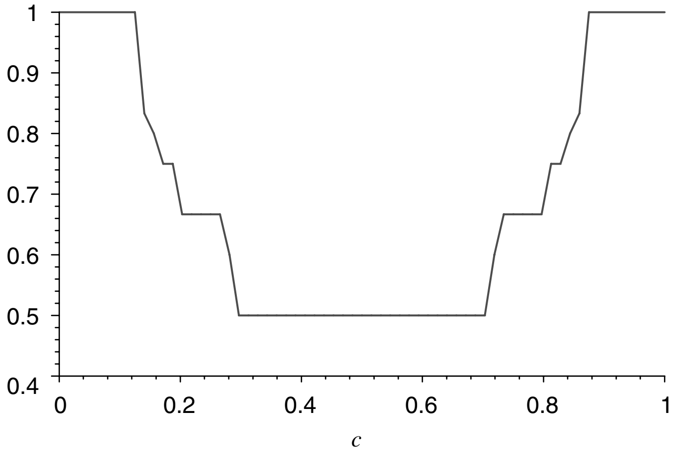

Figure 4. Piecewise-linear approximation to $c \mapsto \alpha(A_{c,1/4})$.

and contains a pre-image of each point on the circle. If $l < 1/2$ then the function $c \mapsto \alpha(A_{c,l})$ is not constant. The case $l = 1/4$ is illustrated in figure 4.

The analysis of maximum hitting frequencies for the full family of closed intervals $A_{c,l}$ is considerably more involved than for the family of centrally symmetric intervals. The chief difficulty is that proposition 3 cannot, in general, be used; in particular, $\alpha(A_{c,l})$ is usually not the reciprocal of a natural number. On the other hand, lemma 4 can be applied to the study of this family, though its exploitation is more delicate. Further details of the map $(c, l) \mapsto \alpha(A_{c,l})$ will appear elsewhere.

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