

Explicit a priori bounds on transfer operator eigenvalues

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ABSTRACT. We provide explicit bounds on the eigenvalues of transfer operators defined in terms of holomorphic data.

Linear operators of the form $\mathcal{L}f = \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$, so-called *transfer operators* (see e.g. [Bal, Rue1, Rue2]), arise in a number of problems in dynamical systems. If the T_i are inverse branches of an expanding map T , and the weight functions w_i are positive, the spectrum of \mathcal{L} has well-known interpretations in terms of the exponential mixing rate of an invariant Gibbs measure (see [Bal]). Applications also arise when the w_i are real-valued (e.g. [CCR, JMS, Pol]) or complex-valued (e.g. [Dol, PS]).

In this article we suppose that T_i and w_i are analytic functions of d variables, for each i in some countable¹ index set \mathcal{I} . Under suitable hypotheses on T_i and w_i the transfer operator \mathcal{L} defines a compact operator on Hardy space $H^2(B)$, and we can give completely explicit bounds on its eigenvalue sequence² $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$:

THEOREM 1. *Suppose there is a complex Euclidean ball $B \subset \mathbb{C}^d$ such that each $w_i : B \rightarrow \mathbb{C}$ is holomorphic with $\sum_{i \in \mathcal{I}} \sup_{z \in B} |w_i(z)| < \infty$, and each $T_i : B \rightarrow B$ is holomorphic with $\cup_{i \in \mathcal{I}} T_i(B)$ contained in the ball concentric with B whose radius is $r < 1$ times that of B .*

Then $\mathcal{L} : H^2(B) \rightarrow H^2(B)$ is compact and

$$|\lambda_n(\mathcal{L})| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}} n^{(d-1)/(2d)} r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}} \quad \text{for all } n \geq 1, \quad (1)$$

where $W := \sup_{z \in B} \sum_{i \in \mathcal{I}} |w_i(z)|$.

If $d = 1$ then

$$|\lambda_n(\mathcal{L})| \leq \frac{W}{\sqrt{1-r^2}} r^{(n-1)/2} \quad \text{for all } n \geq 1. \quad (2)$$

REMARK 2.

- (i) An estimate of the form $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$ for some (undefined) constants $C > 0$, $\theta \in (0, 1)$ is asserted, either implicitly or explicitly, in the work of several authors (e.g. [FR, Fri, GLZ]); the novelty here is that careful derivation of this bound renders explicit the constants C , θ .
- (ii) Using different techniques, the bound $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$ can also be established in the case where B is an arbitrary open subset of \mathbb{C}^d (see [BJ]), though here our expressions for C , θ are more complicated.

EXAMPLE 3. If $\mathcal{L}f(z) = \sum_{n=1}^\infty \left(\frac{1}{n+z}\right)^2 f\left(\frac{1}{n+z}\right)$ (the Perron-Frobenius operator for the Gauss map $x \mapsto 1/x \pmod{1}$), cf. [May]), $B \subset \mathbb{C}$ may be chosen as the open disc of radius $3/2$

¹Subsequent results are new even when \mathcal{I} is finite, but it is convenient to also allow countably infinite \mathcal{I} .

²Precisely, $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$ denotes the sequence of all eigenvalues of \mathcal{L} counting algebraic multiplicities and ordered by decreasing modulus, with the usual convention (see e.g. [Pie, 3.2.20]) that distinct eigenvalues with the same modulus can be written in any order.

centred at the point 1. In this case $W = \sup_{z \in B} \sum_{n=1}^{\infty} |n+z|^{-2} = \sum_{n=1}^{\infty} (n-1/2)^{-2} = \pi^2/2$ and $r = 2/3$, so (2) yields

$$|\lambda_n(\mathcal{L})| \leq \frac{3\pi^2}{2\sqrt{5}} (2/3)^{(n-1)/2} \quad \text{for all } n \geq 1.$$

NOTATION 4. For an open ball $D \subset \mathbb{C}^d$, let $H^\infty(D)$ denote the Banach space consisting of all bounded holomorphic \mathbb{C} -valued functions on D , with norm $\|f\|_{H^\infty(D)} := \sup_{z \in D} |f(z)|$.

Hardy space $H^2(D)$ (see [Kra, Ch. 8.3]) is the $L^2(\partial D, \sigma)$ -closure of the set of those $f \in H^\infty(D)$ which extend continuously to the boundary ∂D , where σ denotes $(2d-1)$ -dimensional Lebesgue measure on ∂D , normalised so that $\sigma(\partial D) = 1$. In particular, $H^2(D)$ is a Hilbert subspace of $L^2(\partial D, \sigma)$ with each element $f \in H^2(D)$ having a natural holomorphic extension to D (see [Kra, Ch. 1.5]).

In the sequel, no generality is lost by taking B in the statement of Theorem 1 to be the unit ball B_1 , and the smaller concentric ball to be B_r , the ball of radius r centred at 0.

If $L : X_1 \rightarrow X_2$ is a continuous operator between Banach spaces then for $k \geq 1$, its k -th approximation number $a_k(L)$ is defined as

$$a_k(L) = \inf\{\|L - K\| \mid K : X_1 \rightarrow X_2 \text{ linear and continuous with } \text{rank}(K) < k\}.$$

The proof of Theorem 1 hinges on the following two lemmas.

LEMMA 5. *If $J : H^2(B_1) \hookrightarrow H^\infty(B_r)$ denotes the canonical embedding, then J and \mathcal{L} are compact and for all $n \geq 1$*

$$|\lambda_n(\mathcal{L})| \leq W \prod_{k=1}^n a_k(J)^{1/n}. \quad (3)$$

PROOF. If $f \in H^2(B_1)$ and $z \in B_r$ then $|f(z)| \leq (2/(1-r))^{d/2}$ by [Rud, Thm. 7.2.5], so $\{f \mid \|f\|_{H^2(B_1)} \leq 1\}$ is a normal family in $H^\infty(B_r)$, hence relatively compact in $H^\infty(B_r)$ by Montel's Theorem (see [Nar, Ch. 1, Prop. 6]), thus J is compact.

Next observe that if $f \in H^\infty(B_1)$ then $f \in H^2(B_1)$ by [Rud, Thm. 5.6.8] and the canonical embedding $\hat{J} : H^\infty(B_1) \hookrightarrow H^2(B_1)$ is continuous of norm 1, because $\sigma(\partial B_1) = 1$. We claim that $\hat{\mathcal{L}}f := \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : H^\infty(B_r) \rightarrow H^\infty(B_1)$. To see this, fix $f \in H^\infty(B_r)$ and note that $w_i \cdot f \circ T_i \in H^\infty(B_1)$ with $\|w_i \cdot f \circ T_i\|_{H^\infty(B_1)} \leq \|w_i\|_{H^\infty(B_1)} \|f\|_{H^\infty(B_r)}$ for every $i \in \mathcal{I}$. But since $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} \leq \sum_{i \in \mathcal{I}} \|w_i\|_{H^\infty(B_1)} \|f\|_{H^\infty(B_r)}$ and $\sum_{i \in \mathcal{I}} \|w_i\|_{H^\infty(B_1)} < \infty$ by hypothesis, we conclude that $\hat{\mathcal{L}}f \in H^\infty(B_1)$ and that $\hat{\mathcal{L}}$ is continuous. Now $|f(T_i(z))| \leq \|f\|_{H^\infty(B_r)}$ for every $z \in B_1$, $i \in \mathcal{I}$, so $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} = \sup_{z \in B_1} |(\hat{\mathcal{L}}f)(z)| \leq \sup_{z \in B_1} \sum_{i \in \mathcal{I}} |w_i(z)| |f(T_i(z))| \leq W \|f\|_{H^\infty(B_r)}$, and hence $\|\hat{\mathcal{L}}\| \leq W$. Now clearly $\mathcal{L} = \hat{J}\hat{\mathcal{L}}J$, so \mathcal{L} is compact, and

$$a_k(\mathcal{L}) \leq \|\hat{J}\hat{\mathcal{L}}\| a_k(J) \leq W a_k(J) \quad \text{for all } k \geq 1, \quad (4)$$

since in general $a_k(L_1 L_2) \leq \|L_1\| a_k(L_2)$ whenever L_1 and L_2 are bounded operators between Banach spaces (see [Pie, 2.2]). Moreover, since \mathcal{L} is a compact operator on Hilbert space, Weyl's inequality (see [Pie, 3.5.1], [Wey]) asserts that $\prod_{k=1}^n |\lambda_k(\mathcal{L})| \leq \prod_{k=1}^n a_k(\mathcal{L})$ for all $n \geq 1$. Together with (4) this yields (3), because $|\lambda_n(\mathcal{L})| \leq \prod_{k=1}^n |\lambda_k(\mathcal{L})|^{1/n}$. \square

LEMMA 6. *If $h_d(k) := \binom{k+d}{d}$ then for all $n \geq 1$,*

$$a_n(J)^2 \leq \sum_{l=k}^{\infty} h_{d-1}(l) r^{2l} \quad \text{where } k \geq 0 \text{ is such that } h_d(k-1) < n \leq h_d(k). \quad (5)$$

PROOF. $H^2(B_1)$ has reproducing kernel $K(z, \zeta) = (1 - (z, \zeta)_{\mathbb{C}^d})^{-d}$ (see [Kra, Thm. 1.5.5]³), where $(\cdot, \cdot)_{\mathbb{C}^d}$ denotes the Euclidean inner product, and $K(z, \zeta) = \sum_{n=1}^{\infty} p_n(z) \overline{p_n(\zeta)}$ whenever $\{p_n\}_{n=1}^{\infty}$ is an orthonormal basis for $H^2(B_1)$, the series converging pointwise for every $(z, \zeta) \in B_1 \times B_1$ (see [Hal, p. 19]).

Define $J_n : H^2(B_1) \rightarrow H^\infty(B_r)$ by $J_n f = \sum_{k=1}^{n-1} (f, p_k) p_k$. If $z \in B_r$ then

$$\begin{aligned} |Jf(z) - J_n f(z)|^2 &= |f(z) - J_n f(z)|^2 = \left| \sum_{k=n}^{\infty} (f, p_k) p_k(z) \right|^2 \\ &\leq \sum_{k=n}^{\infty} |(f, p_k)|^2 \sum_{k=n}^{\infty} |p_k(z)|^2 \leq \|f\|_{H^2(B_1)}^2 \left(K(z, z) - \sum_{k=1}^{n-1} |p_k(z)|^2 \right), \end{aligned}$$

so

$$a_n(J)^2 \leq \sup_{z \in B_r} \left(K(z, z) - \sum_{k=1}^{n-1} |p_k(z)|^2 \right). \quad (6)$$

If $n = 1$ then $k = 0$, in which case (5) follows from (6) since $\sum_{l=0}^{\infty} h_{d-1}(l) r^{2l} = (1 - r^2)^{-d}$. Now define the orthonormal basis $\{p_{\underline{n}} \mid \underline{n} \in \mathbb{N}_0^d\}$ by (cf. [Rud, Prop. 1.4.8, 1.4.9])

$$p_{\underline{n}}(z) = K_{\underline{n}} z^{\underline{n}} \quad (\underline{n} \in \mathbb{N}_0^d),$$

where $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)! \underline{n}!}}$, $\underline{n} = (n_1, \dots, n_d)$, $z^{\underline{n}} = z_1^{n_1} \dots z_d^{n_d}$, $\underline{n}! = n_1! \dots n_d!$, $|\underline{n}| = n_1 + \dots + n_d$.

If $n \geq 2$ then there are $\binom{k+d-1}{d}$ multinomials of degree less than or equal to $k-1$, so

$$a_n(J)^2 \leq \sup_{z \in B_r} \left(K(z, z) - \sum_{|\underline{n}| \leq k-1} |p_{\underline{n}}(z)|^2 \right) = \sup_{z \in B_r} \sum_{l=k}^{\infty} \sum_{|\underline{n}|=l} |p_{\underline{n}}(z)|^2 \leq \sum_{l=k}^{\infty} \frac{(l+d-1)!}{(d-1)! l!} r^{2l}$$

for all $n > \binom{k+d-1}{d}$, because $\sum_{|\underline{n}|=l} \frac{1}{\underline{n}!} |z^{\underline{n}}|^2 \leq \frac{1}{l!} r^{2l}$ for $z \in B_r$ by the multinomial theorem. \square

PROOF OF THEOREM 1. By Lemma 5 it suffices to bound the geometric means $(\prod_{k=1}^n a_k)^{1/n}$, where $a_k := a_k(J)$. From Lemma 6 it follows that

$$a_n^2 \leq \tilde{\alpha}_n \frac{r^{2\tilde{\beta}_n}}{(1-r^2)^d} \quad \text{for all } n \geq 1, \quad (7)$$

where

$$\begin{aligned} \tilde{\alpha}_n &:= h_{d-1}(k) & \text{for } h_d(k-1) < n \leq h_d(k), \\ \tilde{\beta}_n &:= k \end{aligned}$$

because

$$\sum_{l=k}^{\infty} h_{d-1}(l) r^{2l} = h_{d-1}(k) r^{2k} \sum_{l=0}^{\infty} \frac{h_{d-1}(l+k)}{h_{d-1}(k)} r^{2l} \leq h_{d-1}(k) r^{2k} \sum_{l=0}^{\infty} h_{d-1}(l) r^{2l} = h_{d-1}(k) \frac{r^{2k}}{(1-r^2)^d}.$$

Combining (7) with Lemma 5 gives, for all $n \geq 1$,

$$|\lambda_n(\mathcal{L})| \leq W \alpha_n \frac{r^{\beta_n}}{(1-r^2)^{d/2}}, \quad (8)$$

where

$$\alpha_n := \prod_{l=1}^n \tilde{\alpha}_l^{1/(2n)}, \quad \beta_n := \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l.$$

³Note that the extra factor $(d-1)/(2\pi^d)$ appearing in [Kra, Thm. 1.5.5] is due to a different normalisation of the measure σ on ∂B_1 .

To obtain (1) and (2) from (8) we require an upper bound on α_n and a lower bound on β_n . We start with the bounds for α_n . Observe that

$$\tilde{\alpha}_1 = h_{d-1}(0) = 1, \text{ and } \tilde{\alpha}_l \leq d(l-1)^{1-1/d} \text{ for } l \geq 2. \quad (9)$$

To see this note that

$$\frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} = \frac{(d!)^{1-1/d}}{(d-1)!} \left(\frac{\prod_{l=1}^{d-1} (k+l)^d}{\prod_{l=0}^{d-1} (k+l)^{d-1}} \right)^{1/d} = \frac{(d!)^{1-1/d}}{(d-1)!} \prod_{l=1}^{d-1} \left(1 + \frac{l}{k} \right)^{1/d}$$

is decreasing in k , so if $h_d(k-1) < n \leq h_d(k)$ then $\frac{\tilde{\alpha}_l}{(l-1)^{1-1/d}} \leq \frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} \leq \frac{h_{d-1}(1)}{h_d(0)^{1-1/d}} = d$.

The estimate (9) now yields the upper bound

$$\alpha_n = \prod_{i=1}^n \tilde{\alpha}_i^{1/(2n)} \leq \sqrt{d}((n-1)!)^{(d-1)/(2dn)} \leq \sqrt{d} \left(2 \left(\frac{n}{e} \right)^n \right)^{(d-1)/(2dn)} \leq \sqrt{d} n^{(d-1)/(2d)}, \quad (10)$$

where, for $n > 1$, we have used the estimate $(n-1)! \leq 2 \left(\frac{n}{e} \right)^n$ (i.e. $\log(n-1)! \leq \int_{x=2}^n \log x \, dx \leq n \log n - n + \log 2$).

We now turn to the bounds for β_n . If $h_d(k-1) < l \leq h_d(k)$, so that $\tilde{\beta}_l = k$, then $l \leq h_d(k) \leq (d!)^{-1}(k+d)^d$, which implies $\tilde{\beta}_l = k \geq (d!)^{1/d} l^{1/d} - d$. Therefore

$$\beta_n = \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l \geq -d + (d!)^{1/d} \frac{1}{n} \sum_{l=1}^n l^{1/d} > -d + (d!)^{1/d} \frac{d}{d+1} n^{1/d}, \quad (11)$$

where we have used $\sum_{l=1}^n l^{1/d} > \int_{x=0}^n x^{1/d} \, dx = \frac{d}{d+1} n^{1+1/d}$.

Assertion (1) now follows from (8), (10), and (11). Finally, if $d = 1$ then $\beta_n = \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l = \frac{1}{n} \sum_{l=1}^n (l-1) = (n-1)/2$, and (10) becomes $\alpha_n \leq 1$, so substituting into (8) yields (2). \square

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