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Ergodic optimization for countable alphabet subshifts of finite type

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Abstract. Let X be a one-sided subshift of finite type on a countable alphabet, and $T : X \to X$ the shift map. If $f : X \to \mathbb{R}$ is continuous, we provide conditions guaranteeing that f-maximizing measures exist, and are characterised by the condition that their support lies in a certain compact set.

1. *Introduction*

Let X be a one-sided subshift of finite type on a countable alphabet, and $T : X \to X$ the shift map. Let $\mathcal{M}$ denote the set of T-invariant Borel probability measures. If $f : X \to \mathbb{R}$ is continuous, $\mu \in \mathcal{M}$ is called *f-maximizing* if $\int f\, d\mu$ exists and is greater than or equal to $\int f\, dm$ for all $m \in \mathcal{M}$ such that $\int f\, dm$ exists.

The possible non-compactness of X means that in general an f-maximizing measure need not exist. The purpose of this article is to provide conditions under which f-maximizing measures exist, and are characterized by a topological condition: they are precisely those whose support lies in a certain compact set. Results of this kind are already known for various dynamical systems where X is compact, T has some hyperbolicity, and f is sufficiently regular (see e.g. [**B1, B2, CG, CLT, J**]). In particular, if T is the shift map on some compact full shift X, and $f : X \to \mathbb{R}$ has summable variations, there exists a continuous function $\varphi : X \to \mathbb{R}$ such that the f-maximizing measures are precisely those invariant probability measures whose support is contained in the set of maxima of $f + \varphi - \varphi \circ T$ (see [**B2, CG, J**]). In the non-compact setting, however, this is in general not the case. Our main result (see Theorem 6.1 and Corollary 6.2) asserts that if, furthermore, f is bounded above, and also satisfies a certain *oscillation condition*, then the non-compact analogue of the above result does hold.

2. *Preliminaries*

2.1. *Subshifts of finite type.*

Definition 2.1. Our *alphabet* will be a countably infinite set $\mathcal{I}$, which for definiteness we shall take to be the set of strictly positive integers. The *full shift* X on the alphabet $\mathcal{I}$ is the set of all sequences $x = (x_n)_{n=1}^{\infty}$, where each $x_n \in \mathcal{I}$. Define the *shift map* $T : X \to X$ by $(Tx)_n = x_{n+1}$. A matrix $A = (A_{ij})_{i,j=1}^{\infty}$ is an *adjacency matrix* if each $A_{ij} \in \{0, 1\}$, and its associated *subshift of finite type* X_A is defined by

$$X_A = \{x \in X : A_{x_n x_{n+1}} = 1 \text{ for all } n \geq 1\}.$$

Clearly $T(X_A) \subset X_A$, and we simply denote $T|_{X_A}$ by T. We shall always equip X_A with the complete metric $\delta(x, y) = 2^{-\min\{n : x_n \neq y_n\}}$.

A finite word $w \in \mathcal{I}^n$ is *A-admissible* if $A_{w_i w_{i+1}} = 1$ for all $1 \leq i \leq n - 1$. Given $w \in \mathcal{I}^n$ and $x \in X_A$, the *concatenation* wx is the sequence defined by

$$(wx)_i = \begin{cases} w_i & \text{if } 1 \leq i \leq n, \\ x_{i-n} & \text{if } i \geq n + 1. \end{cases}$$

For any integer $n \geq 1$ we define $\Pi_{A,n} : X_A \to \mathcal{I}^n$ by $\Pi_{A,n}(x) = (x_1, \ldots, x_n)$ (projection onto the first n coordinates). If $w \in \mathcal{I}^n$ then the corresponding *cylinder set* in X_A is defined by

$$[w] = [w]_A = \Pi_{A,n}^{-1}(w) = \{x \in X_A : \Pi_{A,n}(x) = w\}.$$

Define

$$\mathcal{I}_A = \{i \in \mathcal{I} : [i]_A \neq \emptyset\},$$

the set of those symbols which actually appear as an entry in some element of X_A. It is not hard to show that $\mathcal{I}_A$ is finite if and only if X_A is compact.

2.2. *Maximizing measures.*

Let $f : X_A \to \mathbb{R}$ be continuous and bounded above. For any Borel probability measure μ on X_A, the integral $\int f\, d\mu \in [-\infty, \infty)$ is well-defined. Let $\mathcal{M}$ denote the collection of T-invariant Borel probability measures on X_A, and define

$$\alpha(f) = \sup_{m \in \mathcal{M}} \int f\, dm.$$

If $\mu \in \mathcal{M}$ satisfies $\int f\, d\mu = \alpha(f)$, then μ is called an *f-maximizing measure*, or simply a *maximizing measure*. Let $\mathcal{M}_{\max}(f)$ denote the set of f-maximizing measures.

2.3. *Function spaces.*

Let $C(X_A)$ denote the space of continuous real-valued functions on X_A, equipped with the topology of uniform convergence on compact subsets. For $\varphi \in C(X_A)$, and $K \subset X_A$ a compact subset, we shall write $\|\varphi\|_{\infty,K} := \sup_{x \in K} |\varphi(x)|$.

The space $CB(X_A)$ of bounded continuous real-valued functions will be equipped with the uniform metric $d(\varphi, \psi) = \sup_{x \in X_A} |(\varphi - \psi)(x)|$, which makes it a Banach space. Note that (for non-compact X_A) this is not the topology induced by $C(X_A)$.

Let $UCB(X_A)$ denote the space of all bounded uniformly continuous real-valued functions. This is a closed subspace of $CB(X_A)$, so is itself a Banach space when equipped with the uniform norm. Let $UCB^\wedge(X_A)$ denote the set of uniformly continuous real-valued functions on X_A which are bounded above. Of course if X_A is compact then the sets $C(X_A)$, $CB(X_A)$, $UCB(X_A)$, and $UCB^\wedge(X_A)$ all coincide.

2.4. *Normal forms.*

Definition 2.2. A function of the form $\varphi - \varphi \circ T$, where $\varphi \in CB(X_A)$, is called a *coboundary*. Two functions f, g which differ by a coboundary are called *cohomologous*, and we write $f \sim g$.

A function $\tilde{f} \sim f$ is a *normal form*† for f if $\tilde{f}^{-1}(\sup \tilde{f})$ contains the support‡ of some T-invariant probability measure. If f has a normal form $\tilde{f}$ then clearly both f and $\tilde{f}$ are bounded above.

If $f \sim g$ then $\int f\,d\mu = \int g\,d\mu$ for all $\mu \in \mathcal{M}$. Therefore $\alpha(f) = \alpha(g)$ and $\mathcal{M}_{\max}(f) = \mathcal{M}_{\max}(g)$. The proof of the following result is straightforward.

LEMMA 2.3. *If the continuous function $f : X_A \to \mathbb{R}$ has a normal form $\tilde{f}$, then*

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \operatorname{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

2.5. *Essentially compact functions.*

Definition 2.4. Let $T : X_A \to X_A$ be surjective. A continuous function $f : X_A \to \mathbb{R}$ is *essentially compact* if there exists $Y \subset X_A$, $\varphi \in CB(X_A)$, and $c \in \mathbb{R}$, such that:

(a) $\widetilde{Y} := \bigcap_{n=0}^{\infty} T^{-n} Y$ is non-empty and compact;

(b) $T(Y) = X_A$;

(c) for each $x \in X_A$,

$$\varphi(x) + c = \sup_{y \in T^{-1}(x)} (f + \varphi)(y) = \sup_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y). \tag{1}$$

Remark 2.5. If f is essentially compact then it is bounded above; indeed $f(y) \le c + \varphi(Ty) - \varphi(y) \le c + (\sup \varphi - \inf \varphi)$ for all $y \in X_A$. In particular, if $S_n f := \sum_{i=0}^{n-1} f \circ T^i$ then $\sup_{x \in X_A} S_n f(x)$ is real for every $n \ge 1$. The constant c in (1) is necessarily equal (see [**JMU**]) to

$$c(f) = \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X_A} S_n f(x).$$

In general, $c(f) \ge \alpha(f)$ and if X_A is compact then $c(f) = \alpha(f)$ (see [**JMU**]).

The following result, proved in [**JMU**], is the reason for introducing the notion of essential compactness.

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† This terminology arises because $\tilde{f}$ is a privileged element in the equivalence class of functions which are cohomologous to f.

‡ Recall that the *support* of a measure μ, denoted $\operatorname{supp}(\mu)$, is by definition the smallest closed subset $Y \subset X_A$ with $\mu(Y) = 1$.

PROPOSITION 2.6. *Let X_A be a subshift of finite type such that $T : X_A \to X_A$ is surjective. If the continuous function $f : X_A \to \mathbb{R}$ is essentially compact, and $\varphi \in CB(X_A)$ is as in Definition 2.4, then $\tilde{f} = f + \varphi - \varphi \circ T$ is a normal form for f and hence*

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \operatorname{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

3. *Nonlinear endofunctions*

Definition 3.1. Let X_A be a subshift of finite type such that $T : X_A \to X_A$ is surjective. If $\varphi : X_A \to \mathbb{R}$ then for each $x \in X_A$, define $M_f\varphi(x) \in (-\infty, \infty]$ by

$$M_f\varphi(x) := \sup_{y \in T^{-1}x} (f + \varphi)(y). \tag{2}$$

LEMMA 3.2. *Let X_A be a subshift of finite type such that $T : X_A \to X_A$ is surjective. If $f \in UCB^\wedge(X_A)$ then M_f defines a (nonlinear) endofunction of $UCB^\wedge(X_A)$.*

Proof. Let $\varphi \in UCB^\wedge(X_A)$. Clearly $M_f\varphi$ is bounded above. Since $M_f\varphi = M_0(f + \varphi)$, it suffices to show that if $g \in UCB^\wedge(X_A)$ then $M_0 g$ is uniformly continuous. Let $\varepsilon > 0$. Choose an integer $n \geq 1$ such that if $u, v \in X_A$ satisfy $\Pi_{A,n+1}(u) = \Pi_{A,n+1}(v)$, then $|g(u) - g(v)| < \varepsilon/2$. Fix $x, y \in X_A$ such that $\Pi_{A,n}(x) = \Pi_{A,n}(y)$. Choose $a \in \mathcal{I}$ such that ax is A-admissible and $g(ax) \leq M_0 g(x) < g(ax) + \varepsilon/2$. Since ay is A-admissible, $g(ay) \leq M_0(g)(y)$, so $-\varepsilon < g(ay) - g(ax) - \varepsilon/2 < M_0 g(y) - M_0 g(x)$. Interchanging x and y gives $-\varepsilon < M_0 g(x) - M_0 g(y)$, so $|M_0 g(x) - M_0 g(y)| < \varepsilon$ and therefore $M_0 g$ is uniformly continuous. □

The endofunction M_f was introduced by Bousch [**B1**] (see also [**B2**]). A closely related object, the Lax–Oleinik semi-group, appears in the work of Fathi [**F**].

In view of Proposition 2.6, we shall henceforth be concerned with finding sufficient conditions for the essential compactness of f. In particular, equation (1) means we will need to show that $M_f\varphi = \varphi + c$ for some $\varphi \in CB(X_A)$. The first step in proving the existence of such a φ is to find conditions under which M_f is an endofunction of the Banach space $UCB(X_A)$. Under these conditions, $M_f : UCB(X_A) \to UCB(X_A)$ has *approximate* fixed points (see Lemma 3.4): for each $0 \leq \lambda < 1$ the equation $\varphi = M_f(\lambda\varphi)$ has a (unique) solution in $UCB(X_A)$, since M_f is 1-Lipschitz for the uniform distance. Later, under extra hypotheses on f and A, it will be shown that if φ is any accumulation point (in a suitable topology) of the family of approximate fixed points, then in fact $M_f\varphi = \varphi + c$. This general strategy is patterned on the proof of [**B2**, Théorème 1], although the non-compactness of X_A complicates matters considerably.

The approximate fixed points for M_f are actual fixed points for the following endofunctions.

Definition 3.3. Let X_A be a subshift of finite type such that $T : X_A \to X_A$ is surjective. Let f be bounded above. For $0 \leq \lambda \leq 1$, define, for any function $\varphi : X_A \to \mathbb{R}$,

$$M_{f,\lambda}\varphi(x) = \sup_{y \in T^{-1}(x)} (f + \lambda\varphi)(y),$$

so that $M_{f,1} = M_f$. Note that if φ is bounded above then so is $M_{f,\lambda}\varphi$. The iterates of $M_{f,\lambda}$ can be expressed as

$$M^n_{f,\lambda}\varphi(x) = \sup\{S_{\lambda,n}f(w_x x) + \lambda^n \varphi(w_x x) : w_x \in \mathcal{I}^n,\, w_x x \in X_A\},$$

where $S_{\lambda,n}f(z) := \sum_{j=0}^{n-1} \lambda^{n-1-j} f(T^j z)$.

LEMMA 3.4. *Let* X_A *be a subshift of finite type. Suppose* f *is uniformly continuous and bounded above, and there exists a subset* $Z \subset X_A$ *such that* $T(Z) = X_A$ *and* $\inf f|_Z > -\infty$.

For each $0 \le \lambda \le 1$, $M_{f,\lambda}$ *defines an endofunction of the Banach space* $UCB(X_A)$.

If $0 \le \lambda < 1$ *then* $M_{f,\lambda}$ *is a contraction on* $UCB(X_A)$, *hence has a unique fixed point* $\varphi_\lambda \in UCB(X_A)$.

Proof. Since $M_{f,\lambda}$ is the composition of M_f and the homothety $\varphi \mapsto \lambda\varphi$, and since $f \in UCB^\wedge(X_A)$, Lemma 3.2 implies that $M_{f,\lambda}$ preserves the space $UCB^\wedge(X_A)$. To show that $M_{f,\lambda}$ preserves $UCB(X_A)$, it remains to check that if $\varphi \in UCB(X_A)$ then $M_{f,\lambda}\varphi$ is bounded below. This is the case because $\inf M_{f,\lambda}\varphi \ge \inf f|_Z + \lambda \inf \varphi > -\infty$.

The operator $M_{f,\lambda} : UCB(X_A) \to UCB(X_A)$ is λ-Lipschitz with respect to the complete metric $d(\varphi, \psi) = \sup_{x\in X_A} |(\varphi - \psi)(x)|$, so for $0 \le \lambda < 1$ it has a unique fixed point $\varphi_\lambda \in UCB(X_A)$. □

4. *Summable variations and primitivity*

Our aim is to extract an accumulation point from the family of approximate fixed points $(\varphi_\lambda)_{0\le\lambda<1}$. To this end it will be useful to prove that the family is equicontinuous (Lemma 4.2), and that its global oscillation is bounded independently of λ (Lemma 4.4). These results will be obtained by imposing further control on the modulus of continuity of the function f, something stronger than uniform continuity.

Definition 4.1. For $n \ge 1$ the nth *variation* of $f : X_A \to \mathbb{R}$ is defined by

$$\mathrm{var}_n(f) = \sup_{\Pi_{A,n}(x)=\Pi_{A,n}(y)} \{f(x) - f(y)\}.$$

Note that f is uniformly continuous if and only if $\mathrm{var}_n(f) \to 0$ as $n \to \infty$.

We say f has *summable variations* if

$$\sum_{n=1}^{\infty} \mathrm{var}_n(f) < \infty.$$

The zeroth variation is defined as $\mathrm{var}_0(f) = \sup_{x,y\in X_A}\{f(x) - f(y)\}$. Note that it is not included in the above sum, so summable variations do not imply boundedness.

LEMMA 4.2. *Let* X_A *be a subshift of finite type. Suppose* f *is bounded above, with summable variations, and there exists a subset* $Z \subset X_A$ *such that* $T(Z) = X_A$ *and* $\inf f|_Z > -\infty$. *Then for all* $0 \le \lambda < 1$, *the fixed point* $\varphi_\lambda \in UCB(X_A)$ *of* $M_{f,\lambda}$ *satisfies*

$$\mathrm{var}_i(\varphi_\lambda) \le \sum_{j=i+1}^{\infty} \mathrm{var}_j(f) \quad \textit{for all } i \ge 1. \tag{3}$$

Proof. Let $\varphi \in UCB(X_A)$. For $j \geq 1$ we shall consider the jth variation of $M_{f,\lambda}\varphi$. Suppose $x, y \in X_A$ satisfy $\Pi_{A,j}(x) = \Pi_{A,j}(y)$. For any $\varepsilon > 0$ we can find $i \in \mathcal{I}$ with $ix \in X_A$ such that

$$M_{f,\lambda}\varphi(x) < \varepsilon + (f + \lambda\varphi)(ix).$$

On the other hand, we have $M_{f,\lambda}\varphi(y) \geq (f + \lambda\varphi)(z)$ for all $z \in T^{-1}(y)$. In particular, we may choose $z = iy$: we know that $iy \in X_A$ because $ix \in X_A$ and $\Pi_{A,1}(x) = \Pi_{A,1}(y)$, and X_A is a subshift of finite type. Therefore,

$$M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) < \varepsilon + (f + \lambda\varphi)(ix) - (f + \lambda\varphi)(iy),$$

from which we deduce that

$$\begin{aligned} M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) &< \varepsilon + \operatorname{var}_{j+1}(f + \lambda\varphi) \\ &\leq \varepsilon + \operatorname{var}_{j+1}(f) + \operatorname{var}_{j+1}(\varphi). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we in fact have

$$\operatorname{var}_j(M_{f,\lambda}\varphi) \leq \operatorname{var}_{j+1}(f) + \operatorname{var}_{j+1}(\varphi),$$

so in particular

$$\operatorname{var}_j(\varphi_\lambda) \leq \operatorname{var}_{j+1}(f) + \operatorname{var}_{j+1}(\varphi_\lambda). \tag{4}$$

Now $\operatorname{var}_{j+1}(\varphi_\lambda) \to 0$ as $j \to \infty$ since φ_λ is uniformly continuous, so iteration of equation (4) yields

$$\operatorname{var}_i(\varphi_\lambda) \leq \sum_{j=i+1}^{\infty} \operatorname{var}_j(f) \quad \text{for all } i \geq 1,$$

which is the required inequality (3). □

We require the following assumption on the matrix A in order to control, uniformly in λ, the zeroth variation of the fixed points φ_λ.

Definition 4.3. An adjacency matrix A, and the corresponding subshift of finite type X_A, are called *primitive* if there exists an integer $N \geq 0$, and a non-empty subset $\mathcal{J} \subset \mathcal{I}$, such that for all $x \in X_A$ and all $i \in \mathcal{I}_A$ there exists $w \in \mathcal{J}^N$ with $iwx \in X_A$. Any such pair $(N, \mathcal{J})$ is called a *primitive pair* for A; N is a *primitive constant*, and $\mathcal{J}$ a *primitive alphabet*. If there exists a finite primitive alphabet $\mathcal{J}$ then we say that both A and X_A are *finitely primitive*.

Note that if A is primitive then $T : X_A \to X_A$ is surjective: for any primitive alphabet $\mathcal{J}$, the set $Z = \Pi_{A,1}^{-1}(\mathcal{J})$ is such that $T(Z) = X_A$.

LEMMA 4.4. *Suppose, in addition to the hypotheses of Lemma 4.2, that X_A is primitive. Then for all $0 \leq \lambda < 1$, the fixed point $\varphi_\lambda \in UCB(X_A)$ of $M_{f,\lambda}$ satisfies*

$$\operatorname{var}_0(\varphi_\lambda) \leq N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})}) + \sum_{j=1}^{\infty} \operatorname{var}_j(f) \tag{5}$$

for any primitive pair $(N, \mathcal{J})$ for X_A.

Proof. Let $m \geq 0$ and write $n = N + m$. For any $\varphi \in UCB(X_A)$, $x \in X_A$, and $\varepsilon > 0$, we can find words $v \in \mathcal{I}^N$ and $u \in \mathcal{I}^m$ such that $uvx \in X_A$ and

$$M^n_{f,\lambda}\varphi(x) < S_{\lambda,n} f(uvx) + \lambda^n \varphi(uvx) + \varepsilon.$$

For any $y \in X_A$, primitivity means we can find $w \in \mathcal{J}^N$ such that $uwy \in X_A$. Clearly

$$M^n_{f,\lambda}\varphi(y) \geq S_{\lambda,n} f(uwy) + \lambda^n \varphi(uwy),$$

and therefore

$$M^n_{f,\lambda}\varphi(x) - M^n_{f,\lambda}\varphi(y) < S_{\lambda,n} f(uvx) - S_{\lambda,n} f(uwy) + \lambda^n[\varphi(uvx) - \varphi(uwy)] + \varepsilon. \quad (6)$$

Now

$$\lambda^n[\varphi(uvx) - \varphi(uwy)] < \mathrm{var}_m(\varphi),$$

and

$$\begin{aligned} S_{\lambda,n} f(uvx) - S_{\lambda,n} f(uwy) &= \sum_{i=0}^{n-1} \lambda^{n-1-i}[f(T^i uvx) - f(T^i uwy)] \\ &\leq \sum_{i=0}^{m-1} \lambda^{n-1-i}\,\mathrm{var}_{m-i}(f) + \sum_{i=m}^{n-1} \lambda^{n-1-i}(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}) \\ &\leq \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary in equation (6), we deduce

$$M^n_{f,\lambda}\varphi(x) - M^n_{f,\lambda}\varphi(y) \leq \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}) + \mathrm{var}_m(\varphi).$$

In particular, choosing φ to be the fixed point φ_λ of $M_{f,\lambda}$ gives

$$\varphi_\lambda(x) - \varphi_\lambda(y) \leq \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}) + \mathrm{var}_m(\varphi_\lambda).$$

However, $\mathrm{var}_m(\varphi_\lambda) \to 0$ as $m \to \infty$, since φ_λ is uniformly continuous, so

$$\mathrm{var}_0(\varphi_\lambda) \leq \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}),$$

as required. □

5. *A condition on the oscillation of f*

The family $(\varphi_\lambda)_{0\leq\lambda<1}$ of approximate fixed points need not itself be uniformly bounded. However, if we define $\varphi^*_\lambda := \varphi_\lambda - \inf \varphi_\lambda$ then $\inf \varphi^*_\lambda = 0$ for each λ, and $\mathrm{var}_0(\varphi^*_\lambda) = \mathrm{var}_0(\varphi_\lambda)$. So provided the right-hand side of inequality (5) is finite†, this inequality implies

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† The right-hand side of inequality (5) is $N(\sup f - \inf f|_{\Pi^{-1}_{A,1}(\mathcal{J})}) + \sum_{j=1}^{\infty} \mathrm{var}_j(f)$, which is finite if for example $\mathcal{J}$ is finite, or if f is bounded, or more generally if f satisfies the oscillation condition introduced in Definition 5.1.

that $(\varphi_\lambda^*)_{0\le\lambda<1}$ *is* bounded independently of λ. It is the family $(\varphi_\lambda^*)_{0\le\lambda<1}$ which will provide an accumulation point.

Although $UCB(X_A)$ was a convenient space in which to find the approximate fixed points φ_λ, it is not an appropriate space in which to find *accumulation points* of the associated family $(\varphi_\lambda^*)_{0\le\lambda<1}$, which *a priori* is not pre-compact in $UCB(X_A)$ (equipped with the uniform distance). On the other hand, $(\varphi_\lambda^*)_{0\le\lambda<1}$ is equicontinuous by inequality (3), and uniformly bounded provided the right-hand side of inequality (5) is finite, in which case the Ascoli–Arzelà theorem implies it is pre-compact in the space $C(X_A)$ (equipped with the topology of uniform convergence on compact subsets). So, provided the right-hand side of inequality (5) is finite, there exists an accumulation point $\varphi_1^* \in C(X_A)$ as $\lambda \nearrow 1$. Indeed $\varphi_1^* \in UCB(X_A)$, since from Lemmas 4.2 and 4.4 it follows that $\mathrm{var}_i(\varphi_1^*) \le \sum_{j=i+1}^{\infty} \mathrm{var}_j(f) \to 0$ as $i \to \infty$, and

$$\mathrm{var}_0(\varphi_1^*) \le N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})}) + \sum_{j=1}^{\infty} \mathrm{var}_j(f) < \infty. \tag{7}$$

For $0 \le \lambda < 1$ the fixed point equation $M_{f,\lambda}\varphi_\lambda = \varphi_\lambda$ is equivalent to

$$M_{f,\lambda}\varphi_\lambda^* = \varphi_\lambda^* + (1-\lambda)\inf \varphi_\lambda,$$

and we would like to use this to show that $M_f\varphi_1^* = \varphi_1^* + c(f)$. This equation would follow readily *if* the endofunction $M_f : UCB(X_A) \to UCB(X_A)$ were continuous with respect to the subspace topology induced by $C(X_A)$. However, *a priori* this is not the case†: continuity in this sense is equivalent to asserting that for every compact subset $K \subset X_A$ there exists another compact subset L such that, for all $\psi \in UCB(X_A)$, the restriction $M_f\psi|_K$ can be expressed as a (continuous) function of $\psi|_L$. In general this is not the case, since $M_f\psi(x)$ is defined by taking a supremum over the (non-compact) set $T^{-1}(x)$.

We therefore require an additional hypothesis on f, a certain quantitative control on its variations. This will ensure that, for any $0 \le \lambda \le 1$, only *finitely many* preimages $y \in T^{-1}(x)$ can contribute to the supremum defining $M_f\varphi_\lambda^*(x)$.

Definition 5.1. Let X_A be a primitive subshift of finite type, and suppose that $f : X_A \to \mathbb{R}$ is bounded above, with summable variations. We say that f satisfies the *oscillation condition* if for some primitive pair $(N, \mathcal{J})$, and some set Z with $T(Z) = X_A$,

$$N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})}) + \sum_{j=1}^{\infty} \mathrm{var}_j(f) < \inf f|_Z - \sup f|_{[i]_A} \tag{8}$$

for all sufficiently large $i \in \mathcal{I}$.

So roughly speaking, the oscillation condition ensures that the values of f on Z are sufficiently larger than its values ‘at infinity’. Of course, a necessary condition for the oscillation condition (8) to hold is that $\inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})} > -\infty$ and $\inf f|_Z > -\infty$. Note that if X_A is the full shift then the oscillation condition can be simplified: we may take $N = 0$

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† By contrast, $M_f : UCB(X_A) \to UCB(X_A)$ is (1-Lipschitz) continuous with respect to the supremum norm. However, this fact cannot be exploited, because we do not know that φ_1^* is an accumulation point of $(\varphi_\lambda^*)_{0\le\lambda<1}$ in this stronger topology.

(and $\mathcal{J}$ finite), and choose Z to be any length-one cylinder set $[I]$, so that the oscillation condition (8) holds when

$$\sum_{j=1}^{\infty} \operatorname{var}_j(f) < \inf f|_{[I]} - \sup f|_{[i]}$$

for all sufficiently large $i \in \mathcal{I}$.

Importantly, if f satisfies the oscillation condition (8) then the right-hand side of inequality (5) is *finite*. This means that we may apply the Ascoli–Arzelà theorem as above to deduce the existence of an accumulation point φ_1^* of the family $(\varphi_\lambda^*)_{0\le\lambda<1}$.

Lemma 5.2. *Let X_A be a primitive subshift of finite type. Suppose $f : X_A \to \mathbb{R}$ has summable variations, is bounded above, and satisfies the oscillation condition. Then there exists $J \in \mathcal{I}$ such that for all* $0 \le \lambda, \mu \le 1$,

$$M_{f,\mu}\varphi_\lambda^*(x) = \max_{y\in T^{-1}(x)\cap\left(\bigcup_{i=1}^{J}[i]_A\right)} (f + \mu\varphi_\lambda^*)(y) \quad \textit{for all } x \in X_A. \tag{9}$$

Proof. The oscillation condition means there is a primitive pair $(N, \mathcal{J})$, and a subset $Z \subset X_A$ with $T(Z) = X_A$, such that the oscillation condition (8) holds for all sufficiently large i. In particular, as noted above, $\inf f|_Z > -\infty$, so we may use Lemmas 4.2 and 4.4. Therefore, the inequalities (5) and (7) hold, and imply that

$$\begin{aligned}
\mu \operatorname{var}_0(\varphi_\lambda^*) &= \mu \operatorname{var}_0(\varphi_\lambda) \\
&\le \operatorname{var}_0(\varphi_\lambda) \\
&\le N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})}) + \sum_{j=1}^{\infty} \operatorname{var}_j(f) \\
&< \inf f|_Z - \sup f|_{[i]_A}
\end{aligned} \tag{10}$$

for all i sufficiently large, $i > J$ say. In particular,

$$\inf f|_Z - \sup f|_{[i]_A} > 0 \quad \text{for all } i > J,$$

so if we define $Y := \bigcup_{i=1}^{J}[i]_A$ then $Z \subset Y$.

Let $x \in X_A$ be arbitrary, and suppose that $ix \in X_A$ for some $i > J$. Now x has at least one preimage $jx \in Z \subset Y$, and by equation (10) we know that

$$\begin{aligned}
f(jx) - f(ix) &> \mu \operatorname{var}_0(\varphi_\lambda^*) \\
&\ge \mu(\varphi_\lambda^*(ix) - \varphi_\lambda^*(jx)).
\end{aligned}$$

That is,

$$(f + \mu\varphi_\lambda^*)(jx) - (f + \mu\varphi_\lambda^*)(ix) > 0,$$

so the supremum $\sup_{y\in T^{-1}(x)}(f + \mu\varphi_\lambda^*)(y) = M_{f,\mu}\varphi_\lambda^*(x)$ must be attained by one of the finitely many preimages of x lying in Y. That is,

$$M_{f,\mu}\varphi_\lambda^*(x) = \sup_{y\in T^{-1}(x)} (f + \mu\varphi_\lambda^*)(y) = \max_{y\in T^{-1}(x)\cap Y} (f + \mu\varphi_\lambda^*)(y)$$

for all $x \in X_A$, as required. □

Notice that for $\lambda = \mu = 1$ the equation (9), asserting that we need only check finitely many preimages $y \in T^{-1}(x)$ in order to compute $M_f\varphi_1^*(x)$, is reminiscent of the definition of essential compactness. Indeed in the proof of Theorem 6.1, once we have established that $M_f\varphi_1^* = \varphi_1^* + c(f)$, it is equation (9) which will provide the important condition (c) of Definition 2.4 from which we deduce that f is essentially compact.

Before that, equation (9) is an important ingredient in the proof of the next lemma. Recall that the need for bounds such as the following estimates (13) and (14) provided the motivation for introducing the oscillation condition.

LEMMA 5.3. *Let X_A be a primitive subshift of finite type. Suppose $f : X_A \to \mathbb{R}$ has summable variations, is bounded above, and satisfies the oscillation condition. Then for every compact subset $K \subset X_A$, and for all $0 \le \lambda, \lambda', \mu, \mu' \le 1$,*

$$\|(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*\|_{\infty,K} \le |\mu - \mu'|\|\varphi_\lambda^*\|_{\infty,L} \tag{11}$$

and

$$\|M_{f,\mu}\varphi_\lambda^* - M_{f,\mu}\varphi_{\lambda'}^*\|_{\infty,K} \le \mu\|\varphi_\lambda^* - \varphi_{\lambda'}^*\|_{\infty,L}, \tag{12}$$

where $L = L(K) := T^{-1}K \cap \left(\bigcup_{i=1}^J [i]_A\right)$ is compact and $J \in \mathcal{I}$ is as in Lemma 5.2.

In particular,

$$\|(M_{f,\lambda} - M_f)\varphi_\lambda^*\|_{\infty,K} \le (1-\lambda)\|\varphi_\lambda^*\|_{\infty,L} \tag{13}$$

and

$$\|M_f\varphi_1^* - M_f\varphi_\lambda^*\|_{\infty,K} \le \|\varphi_1^* - \varphi_\lambda^*\|_{\infty,L}. \tag{14}$$

Proof. Let $Y = \bigcup_{i=1}^J [i]_A$, where $J \in \mathcal{I}$ is as in Lemma 5.2. If $K \subset X_A$ is compact and $x \in K$ then Lemma 5.2 implies we can find $y \in T^{-1}(x) \cap Y \subset T^{-1}K \cap Y =: L$ such that $M_{f,\mu}\varphi_\lambda^*(x) = (f + \mu\varphi_\lambda^*)(y)$. Also $M_{f,\mu'}\varphi_\lambda^*(x) \ge (f + \mu'\varphi_\lambda^*)(y)$, so

$$\begin{aligned}(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*(x) &\le (f + \mu\varphi_\lambda^*)(y) - (f + \mu'\varphi_\lambda^*)(y)\\ &= (\mu - \mu')\varphi_\lambda^*(y)\\ &\le |\mu - \mu'|\|\varphi_\lambda^*\|_{\infty,L}.\end{aligned}$$

Reversing the roles of μ, μ', an analogous argument gives

$$(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*(x) \ge -|\mu - \mu'|\|\varphi_\lambda^*\|_{\infty,L},$$

and since $x \in K$ was arbitrary,

$$\|(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*\|_{\infty,K} \le |\mu - \mu'|\|\varphi_\lambda^*\|_{\infty,L},$$

which is the required inequality (11).

Lemma 5.2 also implies that

$$\begin{aligned}M_{f,\mu}\varphi_\lambda^*(x) - M_{f,\mu}\varphi_{\lambda'}^*(x) &= \max_{y \in T^{-1}(x)\cap Y}(f + \mu\varphi_\lambda^*)(y) - \max_{z \in T^{-1}(x)\cap Y}(f + \mu\varphi_{\lambda'}^*)(z)\\ &\le \max_{w \in T^{-1}(x)\cap Y}((f + \mu\varphi_\lambda^*)(w) - (f + \mu\varphi_{\lambda'}^*)(w))\\ &= \mu \max_{w \in T^{-1}(x)\cap Y}(\varphi_\lambda^* - \varphi_{\lambda'}^*)(w)\\ &\le \mu\|\varphi_\lambda^* - \varphi_{\lambda'}^*\|_{\infty,L},\end{aligned}$$

where $L = T^{-1}K \cap Y$, and inequality (12) follows since $x \in K$ was arbitrary. □

6. *The main result*

With the estimates (13) and (14) in hand we are now ready to prove the main result of this paper, giving sufficient conditions for a function f to be essentially compact, and hence to have a normal form.

THEOREM 6.1. *Let X_A be a primitive subshift of finite type. Suppose $f : X_A \to \mathbb{R}$ has summable variations, is bounded above, and satisfies the oscillation condition.*

Then f is essentially compact, and has a normal form $\tilde{f}$. Consequently

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

Proof. As noted previously, the fixed point equation $M_{f,\lambda}\varphi_\lambda = \varphi_\lambda$ is equivalent to

$$M_{f,\lambda}\varphi_\lambda^* = \varphi_\lambda^* + (1-\lambda)\inf \varphi_\lambda, \tag{15}$$

and we want to let $\lambda \nearrow 1$ in equation (15) in order to show that $M_f\varphi_1^* = \varphi_1^* + c(f)$. If $K \subset X_A$ is compact then

$$\begin{aligned}\|M_{f,\lambda}\varphi_\lambda^* - M_f\varphi_1^*\|_{\infty,K} &\leq \|M_{f,\lambda}\varphi_\lambda^* - M_f\varphi_\lambda^*\|_{\infty,K} + \|M_f\varphi_\lambda^* - M_f\varphi_1^*\|_{\infty,K} \\ &\leq (1-\lambda)\|\varphi_\lambda^*\|_{\infty,L} + \|\varphi_1^* - \varphi_\lambda^*\|_{\infty,L},\end{aligned}$$

by the estimates (13) and (14), where $L = L(K)$ is a compact subset of X_A. Since $\|\varphi_\lambda^*\|_{\infty,L}$ is bounded independently of λ, and φ_1^* is an accumulation point as $\lambda \nearrow 1$ of $(\varphi_\lambda^*)_{0\leq\lambda<1}$ in $C(X_A)$, we see that $M_f\varphi_1^*$ is an accumulation point as $\lambda \nearrow 1$ of $(M_{f,\lambda}\varphi_\lambda^*)_{0\leq\lambda<1}$ in $C(X_A)$.

So $(M_f - Id)(\varphi_1^*)$ is an accumulation point as $\lambda \nearrow 1$ of $((M_{f,\lambda} - Id)(\varphi_\lambda^*))_{0\leq\lambda<1}$ in $C(X_A)$. Since each $(M_{f,\lambda} - Id)(\varphi_\lambda^*)$ is a constant function, and constant functions form a closed subspace of $C(X_A)$, the function $(M_f - Id)(\varphi_1^*)$ is also a constant. This constant must be $c(f) = \lim_n (1/n)\sup_{x\in X_A} S_n f(x)$ (see Remark 2.5), so that

$$M_f\varphi_1^* = \varphi_1^* + c(f). \tag{16}$$

If $Y = \bigcup_{i=1}^J [i]_A$, where $J \in \mathcal{I}$ is as in Lemma 5.2, then setting $\lambda = \mu = 1$ in (9), and combining with (16), gives

$$\varphi_1^*(x) + c(f) = M_f\varphi_1^*(x) = \max_{y\in T^{-1}(x)\cap Y} (f + \varphi_1^*)(y)$$

for all $x \in X_A$, which is precisely condition (c) of Definition 2.4. The fact that Y is a finite union of cylinder sets ensures that conditions (a) and (b) of Definition 2.4 are also satisfied. Therefore, f is essentially compact. Proposition 2.6 then implies that $\tilde{f} := f + \varphi_1^* - \varphi_1^* \circ T$ is a normal form for f, and that

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset. \qquad \square$$

In the case where the subshift of finite type X_A is the full shift, Theorem 6.1 implies the following.

COROLLARY 6.2. *Let X be a full shift. Suppose $f : X \to \mathbb{R}$ is bounded above, has summable variations, and that there exists $I \in \mathcal{I}$ such that*

$$\sum_{j=1}^{\infty} \mathrm{var}_j(f) < \inf f|_{[I]} - \sup f|_{[i]} \tag{17}$$

for all i sufficiently large. Then f is essentially compact, hence has a normal form $\tilde{f}$, hence

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

Proof. As noted after Definition 5.1, the inequality (17) implies the oscillation condition (8), so that f satisfies the oscillation condition. The result follows from Theorem 6.1. □

The oscillation condition asserts that the (finite) infimum of f on some set Z is sufficiently larger than its values 'at infinity'. An extreme example of this is when $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$.

COROLLARY 6.3. *Let X_A be a primitive subshift of finite type. Suppose f has summable variations, that there is a primitive alphabet $\mathcal{J}$ for which $\inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})} > -\infty$, and that $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$.*

Then f is essentially compact, hence has a normal form $\tilde{f}$ and

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

Proof. The left-hand side of (8) is finite, because f is bounded above, with summable variations, and $\inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})} > -\infty$. The set $Z = \Pi_{A,1}^{-1}(\mathcal{J})$ satisfies $T(Z) = X_A$, and $\inf f|_Z > -\infty$, so the right-hand side of the oscillation condition (8) tends to $+\infty$ as $i \to \infty$. Therefore, the oscillation condition (8) holds for all sufficiently large i, so f satisfies the oscillation condition, and the result follows from Theorem 6.1. □

The case where $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$ occurs in particular when the summability condition

$$\sum_{i=1}^{\infty} \exp(\sup f|_{[i]_A}) < \infty \tag{18}$$

holds. This condition arises in the development of the thermodynamic formalism for countable state subshifts of finite type, allowing the definition of the Ruelle operator $\mathcal{L}_f \varphi(x) = \sum_{Ty=x} e^{f(y)} \varphi(y)$. If X_A is *finitely* primitive and f has summable variations then the summability condition (18) is equivalent (cf. [**MU**, Proposition 2.7]) to the finiteness of the topological pressure

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n y = y} \exp\left(\sup_{x \in [\Pi_{A,n}(y)]} \sum_{i=0}^{n-1} f(T^i x) \right),$$

and is a necessary condition for the existence of an invariant Gibbs measure for f. Any f satisfying the summability condition (18) is necessarily bounded above, and if $\mathcal{I}_A = \{i \in \mathcal{I} : [i]_A \neq \emptyset\}$ is infinite then f is unbounded below.

COROLLARY 6.4. *Let X_A be a finitely primitive subshift of finite type. Suppose f has summable variations and satisfies the summability condition (18).*

Then f is essentially compact, hence has a normal form $\tilde{f}$, hence

$$\mathcal{M}_{\max}(f) = \{m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\sup \tilde{f})\} \neq \emptyset.$$

Proof. If $\mathcal{J}$ is a finite primitive alphabet for X_A then $\inf f|_{\Pi_{A,1}^{-1}(\mathcal{J})} > -\infty$. The summability condition (18) implies that $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$, with the convention that $\sup f|_{[i]_A} = -\infty$ whenever $[i]_A$ is empty. The result now follows from Corollary 6.3. □

In particular we deduce the following.

COROLLARY 6.5. *Let X_A be a topologically mixing subshift of finite type. If f has summable variations and an invariant Gibbs measure, then f also has a maximizing measure.*

Proof. As mentioned above, the existence of an invariant Gibbs measure implies the summability condition (18). The existence of an invariant Gibbs measure, together with the fact that X_A is topologically mixing, implies that X_A is finitely primitive (see [**Sa**]). The result then follows from Corollary 6.4. □

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