On sums of powers of inverse complete quotients

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Abstract. For an irrational number x, let x_n denote its n-th continued fraction inverse complete quotient, obtained by deleting the first n partial quotients. For any positive real number r, we establish the optimal linear bound on the sum of the r-th powers of the first n complete quotients. That is, we find the smallest constants $\alpha(r), \beta(r)$ such that $x_1^r + \ldots + x_n^r < \alpha(r)n + \beta(r)$ for all $n \geq 1$ and all irrationals x.

1. Introduction

Every irrational number x has a unique expansion as an infinite continued fraction

$$x = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where each partial quotient a_i is an integer, and $a_i \geq 1$ for $i \geq 1$. Let

$$x_n = [0; a_n, a_{n+1}, a_{n+2}, \ldots] = \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cdots}}}$$

denote the n-th *inverse complete quotient*[1] of x, obtained by deleting the first n partial quotients in its continued fraction expansion. Each x_n is an irrational number in the interval $(0, 1)$.

For any positive real number r, we shall be concerned with the growth of the sum

$$x_1^r + \ldots + x_n^r$$

as $n \to \infty$. The growth is at most linear, and the purpose of this note is to identify the optimal constants $\alpha(r)$ and $\beta(r)$ for which[2]

$$x_1^r + \ldots + x_n^r < \alpha(r)n + \beta(r)$$

for all $n \geq 1$ and all irrationals x.

More precisely, we define $\alpha(r)$ and $\beta(r)$ by

$$\alpha(r) = \sup_{x \in \mathbb{R} \setminus \mathbb{Q}} \limsup_{n \to \infty} \, (x_1^r + \ldots + x_n^r)/n \,,$$

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[1] See e.g. [**2**, §10.9, p. 139] for the definition of the n-th complete quotient as $[a_n; a_{n+1}, a_{n+2}, \ldots]$, the reciprocal of $x_n = [0; a_n, a_{n+1}, a_{n+2}, \ldots]$.

[2] It turns out that $\alpha(r)n + \beta(r)$ is a *strict* upper bound on $x_1^r + \ldots + x_n^r$, though this is not a priori obvious.

and[3]

$$\beta(r) = \sup_{x\in\mathbb{R}\setminus\mathbb{Q}} \sup_{n\in\mathbb{N}} (x_1^r + \ldots + x_n^r) - \alpha(r)\, n\,,$$

and prove:

Theorem 1. *For all irrationals x and all $n \geq 1$,*

$$x_1^r + \ldots + x_n^r < \begin{cases} (n-1)\gamma^r + 1 & \text{if } 0 < r \leq r^* \\ (n+1)/2 & \text{if } r \geq r^*\,, \end{cases} \tag{1}$$

where $\gamma = (\sqrt{5}-1)/2 \approx 0\cdot 618$ and $r^* = -\log 2/\log\gamma \approx 1\cdot 4404$.

These inequalities are sharp, in the sense that $\beta(r) = 1-\alpha(r)$ for all $r > 0$, and

$$\alpha(r) = \begin{cases} \gamma^r & \text{for } 0 < r \leq r^* \\ 1/2 & \text{for } r \geq r^*\,. \end{cases}$$

Remark 1. The novelty in Theorem 1 is that the bound on $x_1^r + \ldots + x_n^r$ holds for *all* irrationals, and is sharp. This should be compared with, on the one hand, trivial sub-optimal bounds such as $x_1^r + \ldots + x_n^r < n$, and on the other hand with the equality

$$\lim_{n\to\infty} \frac{1}{n}(x_1^r + \ldots + x_n^r) = \frac{1}{\log 2}\int_0^1 \frac{t^r}{1+t}\, dt = \frac{1}{\log 2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{r+k} \tag{2}$$

which is valid for *Lebesgue almost every* irrational; this latter result follows from the pointwise ergodic theorem applied to the (ergodic) dynamical system $([0,1], T, G)$, where $T : [0,1] \to [0,1]$ is defined (Lebesgue almost everywhere) by $T(x) = 1/x \pmod 1$, and G denotes Gauss measure $dG(t) = (\log 2)^{-1} dt/(1+t)$ (see e.g. [**1**, p. 174]). For example if $r = 1$ then the righthand side of (2) equals $(\log 2)^{-1} - 1 \approx 0\cdot 44$, which is strictly less than $\alpha(1) = \gamma = (\sqrt{5}-1)/2 \approx 0\cdot 618$.

The key to Theorem 1 is the following result, which will be proved in §2.

Theorem 2. *Let $(x_i)_{i=1}^\infty$ be a sequence of reals in $[0,1]$ such that $x_i(1+x_{i+1}) \leq 1$ for all $i \geq 1$. If $0 < r \leq r^*$ then $x_1^r + \ldots + x_n^r \leq (n-1)\gamma^r + 1$ for all $n \geq 1$.*

2. Proof of Theorems 1 and 2

In order to prove Theorem 2, suppose that $0 < r \leq r^*$, and define $\varphi_r : [0,1] \to \mathbb{R}$ by

$$\varphi_r(x) = \begin{cases} 1 - 2\gamma^r & \text{if } 0 \leq x \leq c_r \\ -x^r & \text{if } c_r \leq x \leq \gamma \\ -\gamma^r & \text{if } \gamma \leq x \leq 1\,, \end{cases}$$

where we set

$$c_r := (2\gamma^r - 1)^{1/r} \geq 0\,.$$

Note in particular that φ_r is non-increasing, is strictly decreasing on $[c_r, \gamma]$, and that its global oscillation is

$$\operatorname{Osc}(\varphi_r) := \sup_{x,y\in[0,1]} |\varphi_r(x) - \varphi_r(y)| = 1 - \gamma^r\,.$$

We first claim that if $x, y \in [0,1]$ satisfy $x(1+y) \leq 1$, then

$$x^r - \gamma^r \leq \varphi_r(y) - \varphi_r(x)\,. \tag{3}$$

If $0 \leq x \leq \gamma$ then (3) clearly holds, because $\varphi_r(x) \leq -x^r$ and $\varphi_r(y) \geq \min\varphi_r = -\gamma^r$.

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[3] It is not immediately obvious that $\beta(r)$ is finite, though this turns out to be the case.

To establish (3) when $\gamma \le x \le 1$, first note that

$$\varphi_r(x) = -\gamma^r\,, \tag{4}$$

and $y \le x^{-1} - 1 \le \gamma^{-1} - 1 = \gamma$. Since φ_r is non-increasing, and $x^{-1} - 1 \in [0, \gamma]$,

$$\varphi_r(y) \ge \varphi_r(x^{-1} - 1) = \min\left(1 - 2\gamma^r, -(x^{-1} - 1)^r\right)\,,$$

and combining with (4) gives

$$\varphi_r(y) - \varphi_r(x) - x^r + \gamma^r \ge \min\left(1 - x^r, 2\gamma^r - x^r - (x^{-1} - 1)^r\right)\,. \tag{5}$$

To prove (3) we must show that the righthand side of (5) is non-negative. Since $1 - x^r \ge 0$, this is the case if and only if $(x^{-1} - 1)^r + x^r \le 2\gamma^r$, and writing $z = x^{-1} - 1$ this is equivalent to proving that

$$z^r + (1+z)^{-r} - 2\gamma^r \le 0 \quad \text{for all } z \in [0,\gamma],\ r \in (0, r^*]\,. \tag{6}$$

Define $F_r(z) := z^r + (1+z)^{-r} - 2\gamma^r$. Since $F_r(\gamma) = 0$, the inequality (6) follows if F_r is increasing on $[0, \gamma]$, and for $0 < r \le 1$ this is the case: its derivative $rz^{r-1} - r(1+z)^{-(r+1)}$ is positive because $z^{r-1} \ge 1$ and $(1+z)^{r+1} > 1$. It remains to prove (6) when $1 < r \le r^*$: in this case $r^{-1}F_r''(z) = (r-1)z^{r-2} + (r+1)(1+z)^{-(r+2)} > 0$, so F_r is convex on $[0, \gamma]$, and therefore (6) follows from the fact that $F_r(0) = 1 - 2\gamma^r \le 0$ and $F_r(\gamma) = 0$.

Now suppose the sequence $(x_i)_{i=1}^\infty$ satisfies $x_i(1 + x_{i+1}) \le 1$ for each $i \ge 1$. From (3),

$$x_1^r + \ldots + x_n^r \le n\gamma^r + \varphi_r(x_{n+1}) - \varphi_r(x_1) \le n\gamma^r + \operatorname{Osc}(\varphi_r) = (n-1)\gamma^r + 1 \tag{7}$$

for all $n \ge 1$, and Theorem 2 is proved.

We wish to use Theorem 2 to deduce Theorem 1, firstly in the case $0 < r \le r^*$. Note that if $(x_i)_{i=1}^\infty$ is the sequence of inverse complete quotients of some irrational, then $x_i \in (0, 1)$, and $x_{i+1} = x_i^{-1} - a_i \le x_i^{-1} - 1$, for each $i \ge 1$. So $(x_i)_{i=1}^\infty$ satisfies the hypotheses of Theorem 2, and therefore (7) holds for $0 < r \le r^*$. In particular, (7) implies that $\alpha(r) \le \gamma^r$. Now the constant sequence $(\gamma, \gamma, \ldots)$, which is the sequence of inverse complete quotients for the number γ, attains the supremum in the definition of $\alpha(r)$, so in fact $\alpha(r) = \gamma^r$. From this, and (7), it follows that $\beta(r) \le 1 - \gamma^r$. But

$$\beta(r) = \sup_{x \in \mathbb{R}\setminus\mathbb{Q}} \sup_{n \in \mathbb{N}} \, (x_1^r + \ldots + x_n^r) - \alpha(r)\, n \ge \sup_{x \in \mathbb{R}\setminus\mathbb{Q}} \, x_1^r - \alpha(r) = 1 - \gamma^r\,,$$

so in fact $\beta(r) = 1 - \gamma^r$.

To show that (7) is always a strict inequality, note that equality firstly implies that $\varphi_r(x_{i+1}) = x_i^r - \gamma^r + \varphi_r(x_i)$ for all $1 \le i \le n$, in particular

$$\varphi_r(x_{n+1}) = x_n^r - \gamma^r + \varphi_r(x_n)\,, \tag{8}$$

and secondly that $\varphi_r(x_{n+1}) - \varphi_r(x_1) = \operatorname{Osc}(\varphi_r) = 1 - \gamma^r$, hence

$$x_{n+1} \in [0, c_r]\,. \tag{9}$$

But now if $x_n \in [0, c_r)$ then (8) means $\varphi_r(x_{n+1}) < c_r^r + 1 - 3\gamma^r = -\gamma^r = \min \varphi_r$, a contradiction, while if $x_n \in [c_r, \gamma]$ then (8) becomes $\varphi_r(x_{n+1}) = -\gamma^r$, hence $x_{n+1} \in [\gamma, 1]$, contradicting (9), and finally if $x_n \in (\gamma, 1)$ then (8) means that $\varphi_r(x_{n+1}) = x_n^r - 2\gamma^r < 1 - 2\gamma^r$, which implies that $x_{n+1} \in (c_r, 1]$, again contradicting (9). So we have shown that x_n does not lie in $[0, 1)$, a contradiction; therefore (7) is a strict inequality when $(x_i)_{i=1}^\infty$ is the sequence of inverse complete quotients of an irrational.

Turning now to the case $r > r^*$, note that $r \mapsto x^r$ is strictly decreasing for every $x \in (0,1)$, so $r \mapsto \limsup_{n\to\infty} (x_1^r + \ldots + x_n^r)/n$ is non-increasing for every $x \in \mathbb{R} \setminus \mathbb{Q}$, and hence

$$\alpha(r) = \sup_{x \in \mathbb{R}\setminus\mathbb{Q}} \limsup_{n\to\infty} \frac{x_1^r + \ldots + x_n^r}{n}$$

is non-increasing in r. In particular,

$$\alpha(r) \le \alpha(r^*) = \gamma^{r^*} = 1/2 \quad \text{for all } r > r^* . \tag{10}$$

Now let $x = [0; 1, 1, 1, 2, 1, 3, \ldots]$ be the irrational whose partial quotients a_i are given by $a_i = 1$ for i odd, and $a_i = i/2$ for i even. Then

$$x_{2i-1} = T^{2i-2}(x) = [0; 1, i, \ldots] \to 1 \quad \text{as } i \to \infty ,$$

whereas

$$x_{2i} = T^{2i-1}(x) = [0; i, \ldots] \to 0 \quad \text{as } i \to \infty ,$$

so that

$$\frac{x_1^r + \ldots + x_n^r}{n} \to \frac{1}{2} \quad \text{as } n \to \infty .$$

Therefore

$$\alpha(r) = \sup_{x \in \mathbb{R}\setminus\mathbb{Q}} \limsup_{n\to\infty} \frac{x_1^r + \ldots + x_n^r}{n} \ge \frac{1}{2} ,$$

and together with (10) we deduce that

$$\alpha(r) = 1/2 \quad \text{for all } r > r^* .$$

To show that $\beta(r) = 1/2$, first note that $\beta(r) = \sup_{x\in\mathbb{R}\setminus\mathbb{Q}} \sup_{n\in\mathbb{N}} \ (x_1^r + \ldots + x_n^r) - \alpha(r)n \ge \sup_{x\in\mathbb{R}\setminus\mathbb{Q}} x_1^r - \alpha(r) = 1 - \alpha(r) = 1/2$. Now $\alpha(r^*) = 1/2 = \beta(r^*)$, and if x is irrational then $x_1^{r^*} + \ldots + x_n^{r^*} - \frac{n}{2} \ < \ \frac{1}{2}$ for $n \ge 1$, so if $r > r^*$ then

$$x_1^r + \ldots + x_n^r - \frac{n}{2} < x_1^{r^*} + \ldots + x_n^{r^*} - \frac{n}{2} \ < \ \frac{1}{2}$$

for $n \ge 1$, thereby establishing the second inequality in (1). Moreover we deduce that

$$\beta(r) = \sup_{x\in\mathbb{R}\setminus\mathbb{Q}} \sup_{n\in\mathbb{N}} \ (x_1^r + \ldots + x_n^r) - \alpha(r)n \le 1/2 ,$$

and hence that $\beta(r) = 1/2$ for all $r > r^*$, thus concluding the proof of Theorem 1.

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References

[1] I. P. Cornfeld, S. V. Fomin & Ya. G. Sinai, *Ergodic theory*, Springer-Verlag, 1982.

[2] G. H. Hardy & E. M. Wright, *An introduction to the theory of numbers (Fifth edition)*, Oxford University Press, 1979.

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