Geometric Barycentres Of Invariant Measures For Circle Maps

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Abstract. *For a continuous circle map T, define the barycentre of any T-invariant probability measure μ to be $b(\mu) = \int_{S^1} z\, d\mu(z)$. The set Ω of all such barycentres is a compact convex subset of $\mathbb{C}$. If T is conjugate to a rational rotation via a Möbius map, we prove Ω is a disc. For every piecewise-onto expanding map we prove that the barycentre set has non-empty interior. In this case, each interior point is the barycentre of many invariant measures, but we prove that amongst these there is a unique one which maximises entropy, and that this measure belongs to a distinguished two-parameter family of equilibrium states. This family induces a real-analytic radial foliation of $int(\Omega)$, centred around the barycentre of the global measure of maximal entropy, where each ray is the barycentre locus of some one-parameter section of the family. We explicitly compute these rays for two examples. While developing this framework we also answer a conjecture of Z. Coelho [6] regarding limits of sequences of equilibrium states.*

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Section 1. Introduction.

Let $T : S^1 \to S^1$ be a continuous map of the circle. The set $\mathcal{M}$ of T-invariant Borel probability measures is non-empty, by the Krylov-Bogolioubov Theorem. Geometrically $\mathcal{M}$ is a simplex, and its extremal points are precisely the ergodic measures. For many circle maps $\mathcal{M}$ has a rich structure, and is hard to describe properly. In particular it may well be infinite-dimensional, allowing rather counter-intuitive properties such as density of extremal points (in which case $\mathcal{M}$ is the so-called Poulsen simplex, see Lindenstrauss et al. [12], Glasner & Weiss [8]). We hope to gain insight into $\mathcal{M}$ by considering finite dimensional projections. In particular, to each measure $\mu \in \mathcal{M}$ we define its *barycentre* (or first Fourier coefficient) to be $b(\mu) = \int_{S^1} z\, d\mu(z)$, the point in the plane describing the average position of μ-almost every orbit of T. This is geometrically the most natural two-dimensional projection. We define $\Omega = b(\mathcal{M})$ to be the set of all such barycentres. If $w \in \Omega$ then we let $\mathcal{M}(w)$ denote the convex set $b^{-1}(w) \subset \mathcal{M}$.

We will study the geometry of Ω, and its relation to $\mathcal{M}$. For certain well-known maps the boundary $\partial\Omega$ has rather pathological differentiability properties (see Bousch [2], Hunt & Ott [10], Jenkinson [11], and also §7, §8). In this article we mainly focus on the *interior* of Ω. We highlight a two-parameter family of measures whose barycentres cover $int(\Omega)$, and study the behaviour of these measures near the boundary $\partial\Omega$.

We begin by proving some basic properties common to all continuous circle maps.

Proposition 1. *Let $T : S^1 \to S^1$ be a continuous circle map.*

(a) The barycentre set Ω is a compact convex subset of the unit disc in $\mathbb{C}$.

(b) If $\mathcal{M}(w)$ is a singleton set containing an ergodic measure then w is an extremal point of Ω.

(c) If w is an extremal point of Ω then any extremal point of $\mathcal{M}(w)$ is ergodic.

Proof. (a) Ω is clearly a subset of the unit disc. We recall (see Walters [27]) that $\mathcal{M}$ is a convex set, compact in the weak* topology. The corresponding properties of Ω follow since the barycentre map b is continuous and affine.

(b) Recall that the extremal points of $\mathcal{M}$ are precisely the ergodic measures. If w is not extremal in Ω then there exist $\mu_1, \mu_2 \in \mathcal{M}$, $\mu_1 \neq \mu_2$, and $\alpha \in (0,1)$, such that $w = \alpha b(\mu_1) + (1-\alpha)b(\mu_2)$. Thus the non-ergodic measure $\alpha\mu_1 + (1-\alpha)\mu_2$ belongs to $\mathcal{M}(w)$.

(c) Let μ be an extremal point of $\mathcal{M}(w)$. We will show that in fact μ is extremal in $\mathcal{M}$, and therefore ergodic.

Suppose, for a contradiction, that we can find distinct measures $\mu_1, \mu_2 \in \mathcal{M}$ and $\alpha \in (0,1)$ such that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2 \ . \tag{1}$$

Applying b to (1) gives $w = \alpha b(\mu_1) + (1-\alpha)b(\mu_2)$. But w is extremal in Ω, so $b(\mu_1) = b(\mu_2) = w$. That is, $\mu_1, \mu_2 \in \mathcal{M}(w)$. So (1) implies μ is not extremal in $\mathcal{M}(w)$, a contradiction. $\square$

The convexity of Ω means it is either a point, a line segment, or has interior. If T is uniquely ergodic then clearly Ω is just a point. This occurs, for example, if T is an irrational rotation - the unique invariant measure is normalised Lebesgue measure, which gives barycentre at 0. In fact even if the rotation is rational then the barycentre set is still the origin - in this case the ergodic measures are each concentrated on single periodic orbits, and each have zero barycentre. The fact that each measure in this one-parameter family is evenly distributed is a rather special property of a pure rotation. More generally one might consider circle diffeomorphisms with rational rotation number. We prove:

Theorem A. *If $T : S^1 \to S^1$ is a diffeomorphism, conjugate to a rational rotation via a Möbius map, then its barycentre set is a disc.*

An interesting open question is to determine exactly which compact convex subsets of the disc can arise as the barycentre set of a circle map. The only way Ω can be the whole disc is if T is the identity, since $b(\mu) \in S^1$ if and only if μ is the Dirac measure concentrated on $b(\mu)$. In fact any segment (the convex hull of an interval $I = \{e^{2\pi i t} : a \leq t \leq b\}$) can arise as a barycentre set. We just take T to be the identity on I, then join up its graph in a monotone way (to ensure $S^1 \setminus I$ contains no T-invariant subsets). Note that such maps are not topologically transitive. If $T : z \mapsto e^{2\pi i \alpha}\overline{z}$ is reflection through the line segment joining $e^{\pi i \alpha}$ and $e^{\pi i(\alpha+1)}$, then Ω is precisely this line segment.

For most of this paper we consider the more complicated dynamics of piecewise-onto expanding maps (defined in §3). We prove the following result.

Theorem B. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map. The barycentre set Ω has non-empty interior.*

For a given $w \in \Omega$ it is natural to consider the entropy of those measures with barycentre w, and to study the concave function $w \mapsto \sup_{\mu \in \mathcal{M}(w)} h(\mu)$. More precisely, for $w \in int(\Omega)$ we solve the problem of identifying those $m \in \mathcal{M}(w)$ such that $h(m) = \sup_{\mu \in \mathcal{M}(w)} h(\mu)$.

Theorem C. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map, with barycentre set Ω. Let $K = [0, 1)$. If $w \in int(\Omega)$ then*

(a) $\mathcal{M}(w)$ contains a unique equilibrium state m_{tf_θ} of a function of the form tf_θ, where $t \geq 0$, $\theta \in K$, and $f_\theta(x) = \cos 2\pi(x - \theta)$,

(b) this equilibrium state m_{tf_θ} is the unique measure of maximal entropy amongst all measures in $\mathcal{M}(w)$,

(c) this maximal entropy $h(m_{tf_\theta})$ is positive and equal to the pressure of the function $t\,(f_\theta\, - \langle w, e^{2\pi i\theta}\rangle)$,

(d) for any $\theta \in K$ this maximal entropy $h(m_{tf_\theta})$ is a strictly decreasing function of $t \geq 0$.

The existence and uniqueness part of Theorem C means the barycentres $b(m_{tf_\theta})$ give a natural and geometrically interesting radial foliation of $int(\Omega)$, which moreover is real-analytic. The rays are the curves $t \mapsto b(m_{tf_\theta})$, which we call *equilibrium rays*, and they are centred around their common intersection point $b(m_0)$, the barycentre of the measure of maximal entropy for T. In §6 we address the problem of accurately computing this radial foliation. We express each equilibrium ray in terms of an adapted coordinate system, the components of which are readily computable in terms of poles of zeta functions. We work with the reciprocals of these zeta functions, and consider their power series expansions. Rapid decay of the coefficients in these expansions means we can work with low degree polynomial approximations and still get accurate results. In §7 and §8 we apply our techniques to two specific examples, the doubling map and the tent map. For the doubling map we also prove (Proposition 13) that the radial foliation is conformal at its centre.

In developing this framework we also consider more general questions regarding the limiting behaviour of equilibrium states of the form m_{tf}. For a general Hölder function f, Coelho [6] conjectured that as $t \to \infty$ the equilibrium states m_{tf} always *converge* in the weak* topology. He conjectured further that this limiting measure always takes a particular form, namely that it is the evenly-weighted centroid of so-called *extremal* measures. In §9 we show this second conjecture is false.

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Section 2. Diffeomorphisms with rational rotation number.

In order to re-state Theorem A more precisely, first recall ([17], page 178) that the Möbius homeomorphisms of S^1 are of the form

$$M_a^\varphi(z) = e^{i\varphi}\frac{z-a}{\bar{a}z-1} \quad \text{and} \quad N_a^\varphi(z) = e^{i\varphi}\frac{az-1}{z-\bar{a}} = e^{i\varphi}/M_{\bar{a}}^\varphi(z),$$

where $a \in \mathbb{C}$ with $|a|<1$, and $\varphi \in [0, 2\pi)$.

Theorem A. *Suppose $T = MRM^{-1} : S^1 \to S^1$, where R is the rational rotation $R(z) = e^{2\pi i p/q}z$, and $M = M_a^\varphi$ or N_a^φ is a Möbius homeomorphism.*

Then the barycentre set for T is a disc D. Its centre is

$$c = e^{i(\varphi+Arg(a))}\left(\frac{|a|-|a|^{2q-1}}{1-|a|^{2q}}\right)$$

and its radius is

$$r = \frac{|a|^{q-1}-|a|^{q+1}}{1-|a|^{2q}}.$$

Proof. We assume $M = M_a^\varphi$, the case $M = N_a^\varphi$ being analogous. The only T-ergodic measures are the images under M of the R-ergodic measures, namely the atomic measures supported on single periodic orbits. Now any extremal point of the barycentre set must be the barycentre of at least one ergodic measure. We will show the barycentre locus of the 1-parameter family of atomic measures is precisely the boundary of D.

Now the 1-parameter family of atomic measures m_t for R is given by

$$m_t = \frac{1}{q}\sum_{j=0}^{q-1} \delta(e^{2\pi i t} e^{2\pi i j/q}) \quad , \quad 0 \le t < 1/q,$$

where $\delta(\cdot)$ denotes Dirac mass. Writing $z = e^{2\pi i t}$ and $\mu = M_*(m_t)$, the corresponding barycentre is

$$b(\mu) = \frac{e^{i\varphi}}{q}\left(\sum_{j=0}^{q-1} \frac{z e^{2\pi i j/q} - a}{\bar{a} z e^{2\pi i j/q} - 1}\right).$$

A calculation involving the relation between the roots and coefficients of a polynomial leads us to the formula

$$b(\mu) = e^{i\varphi}\frac{\bar{a}^{q-1} z^q - a}{\bar{a}^q z^q - 1}.$$

Note in particular that $b(\mu) = B(z^q)$, where B is the Möbius map

$$B(z) = e^{i\varphi}\frac{\bar{a}^{q-1} z - a}{\bar{a}^q z - 1}.$$

The barycentre locus L is the path $b(M_*(m_t))$, $0 \le t < 1/q$. But $z \mapsto z^q$ maps the segment $e^{2\pi i t}$, $0 \le t < 1/q$, to the whole circle S^1. So the barycentre locus is the image of S^1 under the Möbius map B, and hence is itself a circle.

Now we determine the centre and radius of the circle L. Let us write $B = E \circ C$, where $E(z) = e^{i\varphi} z$ and

$$C(z) = \frac{\bar{a}^{q-1} z - a}{\bar{a}^q z - 1}.$$

Then

$$C^{-1}(z) = \frac{z - a}{\bar{a}^q z - \bar{a}^{q-1}}.$$

Suppose w lies on the barycentre locus L. Writing $z = e^{-i\varphi} w$ we have that $C^{-1}(z) \in S^1$, so that

$$|z - a| = |\bar{a}^q z - \bar{a}^{q-1}| = |a|^q |z - 1/\bar{a}|.$$

Now $1/\bar{a}$ lies outside the unit circle, and has the same argument as a. In particular this implies that the centre of the circle $E^{-1}(L)$ lies on the line through the origin of angle $Arg(a)$. Moreover, $E^{-1}(L)$ cuts this line at the two points

$$\left(\frac{|a| + |a|^{q-1}}{1 + |a|^q}\right) e^{iArg(a)} \quad \text{and} \quad \left(\frac{|a| - |a|^{q-1}}{1 - |a|^q}\right) e^{iArg(a)}.$$

A computation gives that $E^{-1}(L)$ has centre at

$$e^{iArg(a)}\left(\frac{|a| - |a|^{2q-1}}{1 - |a|^{2q}}\right)$$

and radius
$$\frac{|a|^{q-1} - |a|^{q+1}}{1 - |a|^{2q}}.$$

Applying E we arrive at the desired expressions for the centre and radius of the barycentre locus L. $\square$

Remark 1. The barycentre set is independent of the numerator p of the rotation number p/q, since as a *set*, a periodic orbit of $R_{p/q}$ is independent of p.

Remark 2. Every interior point of the unit disc can arise as the centre of a barycentre set, by the above construction (though note 0 can only be the centre in the degenerate case where T is the identity map). However, for any given centre, the set of possible radii is discretely parametrised, so that the above construction does not guarantee that all subdiscs of the unit disc can arise as barycentre sets.

Section 3. Expanding Maps.

It will be convenient to use the additive circle $K = [0, 1)$, with addition defined modulo one, and usual distance function ρ. From now on we assume that $T : K \to K$ is a piecewise-onto expanding map.

Definition 1. *We say a continuous map $T : K \to K$ is a piecewise-onto expanding map if there exists a finite number of disjoint open intervals $P_1, \ldots, P_r \subset K$ with $K = \cup_{i=1}^r \overline{P_i}$, and*

(a) there exist $\epsilon > 0$ and $\gamma > 1$ such that for all i, if $x, y \in P_i$ satisfy $\rho(x, y) < \epsilon$, then $\rho(Tx, Ty) > \gamma\rho(x, y)$,

(b) $\overline{TP_i} = K$ for each i.

If $x \in K$ then we let $\mathcal{O}(x)$ denote its forward orbit under T. T has a countable infinity of periodic points, and these are dense in the circle. Let $\mathrm{Fix}(n) = \{x \in K : T^n x = x\}$.

Let $\mathcal{H} = \mathcal{H}(K, \mathbb{R})$ denote the space of all real-valued α-Hölder functions, equipped with the usual Banach norm $\|\cdot\| = \|\cdot\|_\alpha + \|\cdot\|_\infty$. Here
$$\|f\|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha} ,$$
while $\|f\|_\infty = \sup_{x \in K} |f(x)|$ is the uniform norm.

Notation. Given a function f, we write $f^n(x) = \sum_{i=0}^{n-1} f(T^i x)$, and $\Omega(f) = \left\{\int f d\mu : \mu \in \mathcal{M}\right\}$.

Since $\mu \mapsto \int f\, d\mu$ is continuous and affine, and $\mathcal{M}$ is compact and convex, then $\Omega(f)$ is also compact and convex. That is, $\Omega(f)$ is a closed interval contained in the closed interval $[\inf f, \sup f]$.

If $u : K \to \mathbb{R}$ is Borel measurable and bounded, and $a \in \mathbb{R}$, then we say the function $u \cdot T - u + a$ is an *essential coboundary* for T. We call u the *transfer function.* We say two functions $f, g \in \mathcal{H}$ are *essentially cohomologous* if they differ by an essential coboundary.

Given a continuous function $g : K \to \mathbb{R}$ we define its *pressure* $P(g)$ by $P(g) = \sup_{\mu\in\mathcal{M}} \left(h(\mu) + \int g\ d\mu\right)$, where $h(\mu)$ is the entropy of T with respect to μ. An *equilibrium state* for g is a measure $m_g \in \mathcal{M}$ satisfying $h(m_g) + \int g\ dm_g = P(g)$.

If g is Hölder then m_g is unique. Moreover, it is ergodic, fully supported, and has positive entropy (see Walters [27]). In fact the resulting measure-preserving map is an exact endomorphism with Bernoulli natural extension (see Walters [26]). Two Hölder functions have the same equilibrium state if and only if they are essentially cohomologous.

For a fixed $f \in \mathcal{H}$, the map $t \mapsto P(tf)$ is an analytic function of the real variable t. If f is not an essential coboundary then $t \mapsto P(tf)$ is strictly convex. Its derivative is given (see Ruelle [24], Parry & Pollicott [18]) by the formula

$$\frac{d}{dt}P(tf)\Big|_{t=t_0} = \int f\ dm_{t_0 f}\ . \tag{2}$$

Lemma 2. *Let $T : K \to K$ be a piecewise-onto expanding map. If $f \in \mathcal{H}(K,\mathbb{R})$ is not an essential coboundary, then*

$$int(\Omega(f)) = \left\{ \frac{d}{dt}P(tf)\Big|_{t=t_0} : t_0 \in \mathbb{R} \right\}.$$

Proof. Let $D = \left\{\frac{d}{dt}P(tf)|_{t=t_0} : t_0 \in \mathbb{R}\right\}$. From (2) we see that $D \subset \Omega(f)$. Note that D is connected, being the continuous image of $\mathbb{R}$, and in fact the strict convexity of $t \mapsto P(tf)$ means that D is an *open* interval. Thus $D \subset int(\Omega(f))$.

We now show that $\Omega(f) \subset \overline{D}$. The definition of pressure means that for any $\mu \in \mathcal{M}$ we have

$$P(tf) \geq h(\mu) + \int tf\, d\mu. \tag{3}$$

The strict convexity of $t \mapsto P(tf)$ guarantees the existence of the asymptotic derivatives $\lim_{t_0\to\infty} \frac{d}{dt}P(tf)|_{t=t_0}$ and $\lim_{t_0\to-\infty} \frac{d}{dt}P(tf)|_{t=t_0}$. Moreover, from (3) we know the graph of $t \mapsto P(tf)$ lies above the line $t \mapsto h(\mu) + t\int f d\mu$. Comparing asymptotic derivatives gives

$$\lim_{t_0\to\infty} \frac{d}{dt}P(tf)\Big|_{t=t_0} \geq \int f d\mu \geq \lim_{t_0\to-\infty} \frac{d}{dt}P(tf)\Big|_{t=t_0}.$$

Since $\mu \in \mathcal{M}$ was arbitrary, we have $\Omega(f) \subset \overline{D}$. $\square$

Notation. If $a \in \Omega(f)$, we denote $\mathcal{M}(f,a) = \left\{\mu \in \mathcal{M} : \int f\ d\mu = a\right\}$.

Lemma 3. *Let $T : K \to K$ be a piecewise-onto expanding map. Suppose $f \in \mathcal{H}$ is not an essential coboundary for T, and $a \in int(\Omega(f))$. Then*

(a) the equilibrium state $m_{t_0 f}$ belongs to $\mathcal{M}(f,a)$, where $t_0 \in \mathbb{R}$ is the unique value satisfying $a = \frac{d}{dt}P(tf)|_{t=t_0} = \int f\ dm_{t_0 f}$,

(b) $m = m_{t_0 f}$ is the unique measure in $\mathcal{M}(f,a)$ satisfying $h(m) = \sup_{\mu\in\mathcal{M}(f,a)} h(\mu)$,

(c) $h(m_{t_0 f}) = P(t_0(f - a))$,

(d) *the entropy map* $t \mapsto h(m_{tf})$ *is analytic, strictly increasing on* $(-\infty, 0]$ *and strictly decreasing on* $[0, \infty)$.

Proof. (a) is clear by (2) and Lemma 2.

(b) Suppose $\mu \in \mathcal{M}(f, a)$ with $\mu \neq m_{t_0 f}$. The equilibrium state $m_{t_0 f}$ satisfies

$$\int t_0 f \, dm_{t_0 f} + h(m_{t_0 f}) > \int t_0 f \, d\mu + h(\mu) \quad . \tag{4}$$

Now $\mu, m_{t_0 f} \in \mathcal{M}(f, a)$, so $\int t_0 f d\mu = t_0 a = \int t_0 f dm_{t_0 f}$. Thus (4) reduces to $h(m_{t_0 f}) > h(\mu)$.

(c) Since $m_{t_0 f} = m_{t_0(f-a)}$ we have

$$\begin{aligned} P(t_0(f-a)) &= \int t_0(f-a) dm_{t_0(f-a)} + h(m_{t_0(f-a)}) \\ &= \int t_0(f-a) dm_{t_0 f} + h(m_{t_0 f}) \\ &= h(m_{t_0 f}). \end{aligned}$$

(d) Analyticity follows from the analyticity of pressure and the relation

$$\begin{aligned} h(m_{t_0 f}) &= P(t_0 f) - t_0 \int f \, dm_{t_0 f} \\ &= P(t_0 f) - t_0 \frac{d}{dt} P(tf) \Big|_{t=t_0} . \end{aligned}$$

A calculation gives

$$\frac{d}{dt} h(m_{tf}) \Big|_{t=t_0} = -t_0 \frac{d^2}{dt^2} P(tf) \Big|_{t=t_0} ,$$

and the result follows from strict convexity of pressure. $\square$

Section 4. The barycentre set has interior.

In this section we prove (Theorem B) that if T is a piecewise-onto expanding map then its barycentre set has non-empty interior, or equivalently that not every barycentre lies on the boundary $\partial\Omega$. We parametrise $\partial\Omega$ by $\theta \in K$, which we think of as indexing a normal to $\partial\Omega$ in the $2\pi\theta$ direction.

We let $q(\theta) = \sup_{w\in\Omega}\langle w, e^{2\pi i\theta}\rangle$ denote the maximal component in the $2\pi\theta$ direction, and $w(\theta) = \{w \in \Omega : \langle w, e^{2\pi i\theta}\rangle = q(\theta)\}$ the set of barycentres achieving this maximum. Here we are identifying $\mathbb{C}$ with $\mathbb{R}^2$, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^2$.

Clearly any $w(\theta)$ must lie in $\partial\Omega$, and is either a point or a line segment, and conversely any point of $\partial\Omega$ must belong to some $w(\theta)$.

We remark that $q(\theta) = \sup_{\mu\in\mathcal{M}}\int f_\theta d\mu$, where $f_\theta : K \to \mathbb{R}$ is defined by $f_\theta(x) = \cos 2\pi(x-\theta) = \langle e^{2\pi ix}, e^{2\pi i\theta}\rangle$. This motivates the following definition.

Definition 2. *Given $f \in \mathcal{H}(K,\mathbb{R})$ we define its optimal ergodic average $Q(f)$ by*

$$Q(f) = \sup_{\mu\in\mathcal{M}} \int f\ d\mu = \sup\ \Omega(f).$$

A measure $m \in \mathcal{M}$ is called f-optimal if $\int f dm = Q(f)$.

The weak* compactness of $\mathcal{M}$ guarantees that an optimal measure always exists, though it need not be unique. By the same proof as used in Lemma 2 we know that

$$Q(f) = \lim_{t_0\to\infty} \int f\ dm_{t_0 f} = \lim_{t_0\to\infty} \frac{d}{dt}P(tf)\bigg|_{t=t_0}. \tag{5}$$

Thus any weak* limit point of m_{tf}, as $t \to \infty$, is an optimal measure. Bousch [3] has a proof that a *generic* $f \in \mathcal{H}(K,\mathbb{R})$ has a unique optimal measure (in fact his argument holds for a wider class of f and T than considered here). In this case the optimal measure is precisely $\lim_{t\to\infty} m_{tf}$.

The following type of lemma was first proved by Mañé [15] in the context of Lagrangian systems. Contreras, Lopes and Thieullen [7] proved the same result for orientation-preserving $C^{1+\epsilon}$ expanding circle maps. The result for piecewise-onto expanding maps is essentially contained in Bousch [2]. Corollary 5 was already known (see Coelho [6]).

Lemma 4. *Let $T : K \to K$ be a piecewise-onto expanding map, and suppose $f \in \mathcal{H}(K,\mathbb{R})$. A measure $\mu \in \mathcal{M}$ is f-optimal if and only if f is cohomologous to the optimal ergodic average $Q(f)$ on the support of μ.*

Corollary 5. *Let $T : K \to K$ be a piecewise-onto expanding map. Suppose $f \in \mathcal{H}(K,\mathbb{R})$ is not an essential coboundary for T. Any optimal measure for f cannot be fully supported.*

Proof of Theorem B. Now $Q(f_\theta) = \sup_{\mu\in\mathcal{M}}\int f_\theta\ d\mu = q(\theta)$, the maximal component in the $2\pi\theta$ direction. Thus $b(\mu) \in \partial\Omega$ if and only if μ is f_θ-optimal for some $\theta \in K$.

Now if Ω had empty interior then every barycentre would lie on $\partial\Omega$, including the barycentre of the measure of maximal entropy m_0. So m_0 would be f_θ-optimal for some

$\theta \in K$. But m_0 is fully supported, so by Corollary 5, f_θ must be an essential coboundary. That is,

$$f_\theta - a = u \cdot T - u \tag{6}$$

for some bounded measurable function $u : K \to \mathbb{R}$ and $a \in \mathbb{R}$. Then a must be the value of f_θ at the fixed point(s) of T, so the left hand side of (6) has exactly 2 zeros. We will argue that the right hand side has at least 3 zeros.

This is immediate in the case when T has at least 3 fixed points. If T has either one or two fixed points, we first use the Livsic regularity theorem [13], [14] (see also [19], [25]) to show that the transfer function u is (Hölder) continuous. We then use the following intermediate value argument to guarantee the remaining zeros.

First suppose T has just one fixed point. Then T must be of topological degree two, and in particular is orientation-preserving. The fixed point is a zero of $u \cdot T - u$, so it remains to find two more zeros.

Whenever we say a point is a *maximum* of u we will mean that u achieves its global maximum at that point. Suppose x_0 is a maximum of u and is not the fixed point (if the maximum is unique and is the fixed point then we just apply an analogous argument to the global minimum). Then $Tx_0 \neq x_0$, and $T^{-1}x_0 = \{x_1, x_2\}$, where $x_1 < x_0 < x_2$. We will treat eight separate cases.

If x_1 and x_2 are maxima of u then they are both zeros of $u \cdot T - u$, and we are done.

If neither of x_1, x_2 is a maximum, and Tx_0 is not a maximum, we obtain the inequalities

$$(v \cdot T - v)(x_1) > 0 > (v \cdot T - v)(x_0) \quad \text{and} \quad (v \cdot T - v)(x_2) > 0 > (v \cdot T - v)(x_0).$$

The intermediate value theorem then gives zeros in both the intervals (x_1, x_0) and (x_0, x_2), and we are done.

If neither of x_1, x_2 is a maximum, but both Tx_0 and T^2x_0 are maxima, then both x_0 and Tx_0 are zeros for $u \cdot T - u$, and we are done.

If neither of x_1, x_2 is a maximum, but Tx_0 is a maximum, but T^2x_0 is not a maximum, then x_0 is a zero of $u \cdot T - u$, so it remains to find one more zero. Let $x_0' \in T^{-1}Tx_0$ satisfy $x_0' \neq x_0$. Then either $x_0 < Tx_0 < x_0'$, or $x_0' < Tx_0 < x_0$. If $v(x_0') = v(Tx_0)$ then x_0' is a zero of $u \cdot T - u$, and we are done. If not, then we have

$$(v \cdot T - v)(x_0') > 0 > (v \cdot T - v)(Tx_0),$$

so the intermediate value theorem guarantees a zero between x_0' and Tx_0, and the previous inequalities imply this zero is not the point x_0. So we are done.

If x_1 is a maximum, but x_2 and Tx_0 are not maxima, then x_1 is a zero of $u \cdot T - u$, and similarly to previous cases we find another zero between x_0 and x_2, so we are done.

An analogous argument applies if x_2 is a maximum, but x_1 and Tx_0 are not maxima.

If x_1 and Tx_0 are maxima, but x_2 is not a maximum, then x_1 and x_0 are both zeros of $u \cdot T - u$, and we are done.

An analogous argument applies if x_2 and Tx_0 are maxima, but x_1 is not a maximum.

This completes the case where T has a single fixed point.

Now suppose T has exactly two fixed points. These fixed points are both zeros of $u \cdot T - u$, so it remains to find a third zero. The above arguments are easily modified to suit this case *unless* u has a unique global minimum, a unique global maximum, and these are attained precisely at the fixed points. In such a case we first note that one of the fixed points must lie on the boundary of the partition. So without loss of generality, let us suppose 0 is the fixed point, and the partition is given by $0 = a_0 < a_1 < \ldots < a_r = 1$. In particular note that $Ta_i = 0$ for $i = 0, \ldots, r$. Let $x_0 \neq 0$ be the other fixed point. Without loss of generality suppose x_0 is the maximum of u and 0 is the minimum. We can choose some pre-image $a_j \neq 0$ of 0, and some pre-image $x_1 \neq x_0$ of x_0 such that neither 0 nor x_0 lies in between a_j and x_1. Then $(v \cdot T - v)(x_1) > 0 > (v \cdot T - v)(a_j)$, so that $v \cdot T - v$ has a zero between a_j and x_1, and this zero is not one of the fixed points. $\square$

Section 5. Equilibrium rays.

Definition 3. *For a fixed $\theta \in K$ we call the curve $r_\theta : t \mapsto b(m_{tf_\theta})$, $t \geq 0$, an equilibrium ray. We call*

$$\begin{aligned} B : \mathbb{R}_{\geq 0} \times K &\to \mathbb{C} \\ (t,\theta) &\mapsto b(m_{tf_\theta}) \end{aligned}$$

the radial map.

Note by Lemma 3 (d) that the entropy of the corresponding equilibrium states m_{tf_θ} is strictly decreasing along each ray.

Proposition 6. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map. The radial map $B : (t, \theta) \mapsto b(m_{tf_\theta})$ is real-analytic.*

Proof. We just note that m_{tf_θ} is a fixed point for the dual of the normalised transfer operator corresponding to the function tf_θ (see Parry & Pollicott [18]). Since the eigenvalue 1 is simple and isolated, it follows by standard perturbation theory (see Appendix V of [18]), that both the eigenvalue and the eigenvector are analytic functions of the parameters. So fixing θ, for example, we have that m_{tf_θ} is a weak* analytic function of t. That is, for any continuous function g the map $t \mapsto \int g \, dm_{tf_\theta}$ is analytic. Similarly if we fix t then $\theta \mapsto \int g \, dm_{tf_\theta}$ is analytic.

Since $b(m_{tf_\theta}) = \int z \, dm_{tf_\theta}(z)$, then choosing $g(z) = z$ gives the result. $\square$

Corollary 7. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map. Each of the equilibrium rays r_θ is real-analytic.* $\square$

Clearly all equilibrium rays r_θ share the same basepoint $r_\theta(0) = b(m_0)$, the barycentre of the unique measure of maximal entropy for T. The substance of the following Proposition 8 is that this is the *only* point where distinct rays intersect. However (see §7, §8) there are examples where rays intersect at infinity. That is, distinct rays can land at the same point on $\partial\Omega$.

Proposition 8. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map. Then the restriction of the radial map B to $\mathbb{R}^+ \times K$ is injective. That is, no equilibrium ray self-intersects, and two distinct rays can only intersect at their common basepoint.*

Proof. Suppose $(t, \theta) \neq (t', \theta')$. We claim the functions tf_θ, $t'f_{\theta'}$ are not essentially cohomologous with respect to T. To see this, suppose for a contradiction that

$$tf_\theta = t'f_{\theta'} + u \cdot T - u + a$$

for some $a \in \mathbb{R}$ and bounded measurable function u. Then

$$t \int f_\theta d\mu - t' \int f_{\theta'} d\mu = a$$

for all $\mu \in \mathcal{M}$. That is,

$$\langle b(\mu), te^{2\pi i\theta} - t'e^{2\pi i\theta'} \rangle \;=\; t\langle b(\mu), e^{2\pi i\theta} \rangle - \; t'\langle b(\mu), e^{2\pi i\theta'} \rangle \;=\; a$$

for all $\mu \in \mathcal{M}$. So all barycentres project to the same value a in the $te^{2\pi i\theta} - t'e^{2\pi i\theta'}$ direction. This is a contradiction, since by Theorem B we know that Ω has an interior. Since tf_θ, $tf_{\theta'}$ are not essentially cohomologous, the corresponding equilibrium states m_{tf_θ}, $m_{t'f_{\theta'}}$ are not equal (see Parry & Pollicott [18], Prop. 3.6).

Now suppose, for a contradiction, that

$$b(m_{tf_\theta}) = w = b(m_{t'f_{\theta'}}). \tag{7}$$

Projecting w onto the $2\pi\theta$ direction gives

$$\int f_\theta \; dm_{tf_\theta} = \int f_\theta \; dm_{t'f_{\theta'}} \;=\; \langle w, e^{2\pi i\theta} \rangle \;=: c \;,$$

so that m_{tf_θ} and $m_{t'f_{\theta'}}$ both belong to $\mathcal{M}(f_\theta, c)$. By Lemma 3 (b) we know that $m = m_{tf_\theta}$ is the unique measure in $\mathcal{M}(f_\theta, c)$ satisfying $h(m) = \sup_{\mu \in \mathcal{M}(f_\theta, c)} h(\mu)$. Thus $h(m_{tf_\theta}) > h(m_{t'f_{\theta'}})$.

If instead we project (7) onto the $2\pi\theta'$ direction, a similar argument gives $h(m_{t'f_{\theta'}}) > h(m_{tf_\theta})$, giving the required contradiction. $\square$

Proposition 9. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map. The radial map*

$$\begin{aligned} B : \mathbb{R}_{\geq 0} \times K &\to \mathbb{C} \\ (t,\theta) &\mapsto b(m_{tf_\theta}) \end{aligned}$$

is a surjection onto the interior of the barycentre set Ω.

Proof. By Proposition 6 we know each equilibrium ray r_θ is a real-analytic, and hence continuous, curve. Note that $r_\theta(0) = b(m_0)$ is the barycentre of the measure of maximal entropy of T, which lies in the interior of Ω.

By (5) we know that

$$\int f_\theta \, dm_{tf_\theta} \to Q(f_\theta) = q(\theta) \qquad \text{as } t \to \infty.$$

This means that the component of $b(m_{tf_\theta})$ in the $2\pi\theta$ direction is converging to the maximal component in the $2\pi\theta$ direction. Thus $b(m_{tf_\theta})$ is accumulating at the boundary of Ω.

So each $R_\theta = \{b(m_{tf_\theta}) : t \geq 0\}$ is a half-open curve in Ω, with closed endpoint at $b(m_0)$ and open endpoint accumulating at $\partial\Omega$.

Now the family r_θ is continuous in $\theta \in K$, by Lemma 6. Moreover, Ω is convex and thus contractible, so the family of curves R_θ sweeps out the whole of $int(\Omega)$. That is,

$$\bigcup_{\theta \in K} R_\theta = int(\Omega). \quad \Box$$

Of course we could have defined rays in terms of any one-parameter family of functions rather than the particular family $f_\theta(x) = \cos 2\pi(x - \theta)$. However in general the properties of Propositions 8 and 9 would not then hold. Such rays might self-intersect, or intersect each other, and need not cover the whole interior of the barycentre set.

By Proposition 8, the restriction of $B : \mathbb{R}_{\geq 0} \times K \to \mathbb{C}$ to $\mathbb{R}^+ \times K$ is injective, while the set $\{0\} \times K$ is mapped to a single point. The same is true for the natural projection $A : \mathbb{R}_{\geq 0} \times K \to \mathbb{C}$ defined by $(t, \theta) \mapsto te^{2\pi i\theta}$. The image of B is $int(\Omega)$, while the image of A is all of $\mathbb{C}$. Hence the homeomorphism $\hat{B} : \mathbb{C} \to int(\Omega)$ satisfying $\hat{B} \circ A = B$ is well-defined and real-analytic. We call it the *induced radial map*.

Proposition 10. *Let $T : S^1 \to S^1$ be a piecewise-onto expanding map, with barycentre set Ω. Let $f_\theta(x) = \cos 2\pi(x - \theta)$, and let m_{tf_θ} denote the equilibrium state of tf_θ. Then the induced radial map $\hat{B}$ is a real-analytic homeomorphism of $\mathbb{C}$ onto the interior of Ω.*

Proof. The result follows immediately from Propositions 6, 8, and 9. $\Box$

Alternatively we can think of the family of equilibrium rays r_θ as a *radial foliation* of $int(\Omega)$, in the sense that their only point of intersection is $b(m_0)$. We are now in a position to answer our question about the measures of maximal entropy in each barycentre class.

Proof of Theorem C. (a) Existence and uniqueness follow from Proposition 10.

(b) Now suppose $\mu, m_{tf_\theta} \in \mathcal{M}(w)$, where $\mu \neq m_{tf_\theta}$. This implies that

$$\mu, m_{tf_\theta} \in \mathcal{M}(f_\theta, \langle w, e^{2\pi i\theta} \rangle).$$

But then Lemma 3 (b) implies that $h(m_{tf_\theta}) > h(\mu)$.

(c) By Lemma 3 (c) we have $h(m_{tf_\theta}) = P(t\ (f_\theta\ - \langle w, e^{2\pi i\theta} \rangle))$, which is positive since m_{tf_θ} is an equilibrium state.

(d) The fact that entropy is strictly decreasing follows from Lemma 3 (d). $\Box$

Section 6. Computing equilibrium rays.

In §5 we introduced a certain real-analytic radial foliation of the interior of Ω by barycentres of equilibrium states m_{tf_θ} which maximise entropy in their barycentre class. In this section we consider the problem of accurately computing the leaves of this foliation. One possible approach is to weak* approximate the equilibrium states m_{tf_θ} by the atomic invariant measures $m^n_{tf_\theta}$ defined by

$$m^n_{tf_\theta} = \left[\sum_{x\in Fix(n)} \exp(tf^n_\theta(x))\right]^{-1} \sum_{x\in Fix(n)} \exp(tf^n_\theta(x))\delta_x$$

for some sufficiently large n, and compute the barycentres for these measures instead. Unfortunately with this method the convergence to the true equilibrium rays is rather slow. We prefer the following more efficient method using (Ruelle) zeta functions and Fredholm determinants, where the convergence is exponentially and super-exponentially fast respectively. In §7 and §8 we implement this approach for the doubling map and the tent map.

Notation. *For $\theta \in K$ we introduce the orthogonal unit vectors $\hat\imath_\theta = \cos 2\pi\theta + i\sin 2\pi\theta$ and $\hat\jmath_\theta = -\sin 2\pi\theta + i\cos 2\pi\theta$, which we refer to as θ coordinates. For an equilibrium state m_{tf_θ} it will be convenient to express its barycentre $b(m_{tf_\theta})$ with respect to these coordinates. We write*

$$r_\theta(t) = b(m_{tf_\theta}) = I(t,\theta)\hat\imath_\theta + J(t,\theta)\hat\jmath_\theta. \tag{8}$$

where $I(t,\theta) = \int f_\theta\, dm_{tf_\theta}$, $J(t,\theta) = \int g_\theta\, dm_{tf_\theta}$ are the components of $b(m_{tf_\theta})$ in the $\hat\imath_\theta$ and $\hat\jmath_\theta$ directions respectively, and $g_\theta(x) = \sin 2\pi(x-\theta)$.

Let us define

$$F(z,t,s) = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}(n)} e^{(tf_\theta + sg_\theta)^n(x)}\right] \tag{9}$$

for $z \in \mathbb{C}$, $s,t \in \mathbb{R}$.

Note that F is just the reciprocal of the weighted Ruelle zeta function (see Ruelle [23]) for the function $tf_\theta + sg_\theta$.

If our map T is differentiable, we can also define the following function (we also call it F since the following two lemmas are valid for either definition):

$$F(z,t,s) = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}(n)} \frac{e^{(tf_\theta + sg_\theta)^n(x)}}{|1-1/(T^n)'(x)|}\right]. \tag{10}$$

This function F is a generalised Fredholm determinant (see for example [1]). In the case where T is piecewise-analytic it is a Fredholm determinant in the sense of Grothendieck [9], and in particular is an entire function of z (see [23]).

Lemma 11. *(Ruelle [22]) Fix $t, s \in \mathbb{R}$, and let $z(t,s)$ denote the zero of $F(z,t,s)$ with smallest absolute value. Then $z(t,s) = e^{-P(tf_\theta + sg_\theta)}$. This zero is simple and isolated.*

Lemma 12. *The components $I(t,\theta)$ and $J(t,\theta)$ are given by*

$$I(t,\theta) = \frac{1}{z(t,0)} \frac{D_2 F(z(t,0),t,0)}{D_1 F(z(t,0),t,0)},$$

$$J(t,\theta) = \frac{1}{z(t,0)} \frac{D_3 F(z(t,0),t,0)}{D_1 F(z(t,0),t,0)},$$

where $D_i F$ denotes the partial derivative with respect to the i^{th} coordinate.

Proof. We calculate $J(t,\theta)$; the computation for $I(t,\theta)$ is similar. From Parry & Pollicott [18], page 60, we have

$$J(t,\theta) = \int g_\theta \, dm_{tf_\theta} = \frac{\partial}{\partial s} P(tf_\theta + sg_\theta) \Big|_{s=0}. \tag{11}$$

From Lemma 11 we have

$$\begin{aligned} \frac{\partial}{\partial s} P(tf_\theta + sg_\theta) \Big|_{s=0} &= \frac{\partial}{\partial s} - \log z(t,s) \Big|_{s=0} \\ &= \frac{-1}{z(t,s)} \frac{\partial z(t,s)}{\partial s} \Big|_{s=0}. \end{aligned} \tag{12}$$

Now $F(z(t,s),t,s) = 0$ for all $t, s \in \mathbb{R}$, so implicit differentiation gives

$$\frac{\partial z(t,s)}{\partial s} \Big|_{s=0} = \frac{-D_3 F(z(0,t),t,0)}{D_1 F(z(0,t),t,0)}. \tag{13}$$

Substituting (13) into (12), and then into (11), gives the result. □

Using formula (8) together with Lemma 12 gives a practical method of computing the equilibrium rays. We expand F as a power series in z,

$$F(z,t,s) = 1 + \sum_{n=1}^{\infty} a_n(t,s) z^n.$$

If we define F by (9) then the coefficients a_n in this power series converge to zero exponentially fast (see Ruelle [23]). In the case where T is piecewise-analytic (as in §7 and §8), and we define F by (10), then the decay of the a_n is at the super-exponential rate $O(\theta^{n^2})$, for some $0 < \theta < 1$ [23]. Hence the leading zero $z(t,s)$ of F is (super-)exponentially-well approximated by the smallest zero $z_N(t,s)$ of the N^{th} order polynomial truncation $F_N(z,t,s) = 1 + \sum_{n=1}^{N} a_n(t,s) z^n$. Replacing F and $z(t,s)$ in the formulae of Lemma 12 by the computable quantities F_N and $z_N(t,s)$ for a suitable choice of N, and then substituting into (8), we obtain an accurate approximation of the ray.

Section 7. The doubling map.

In this section we concentrate on the most familiar example of a piecewise-onto expanding map, the angle-doubling map $T : z \mapsto z^2$, and make some more precise remarks about its family of equilibrium rays. Most of what we say is valid, with suitable modifications, for any of the standard covering maps $z \mapsto z^k$, $k \geq 2$ (see Jenkinson [11]). It has recently been proved (see Bousch [2]) that the boundary of the barycentre set Ω corresponds to a certain one-parameter family of ergodic measures known as Sturmian measures (for full details of the properties of $\partial\Omega$ see Bousch [2], Jenkinson [11]). These measures are either atomic or are supported on a Cantor set of zero Hausdorff dimension, and are characterised by the property that their support is contained in a closed semi-circle (see Bullett & Sentenac [5], Morse & Hedlund [16]). The barycentre set is *strictly* convex, so that its boundary contains no line segments. Thus every boundary point is *extremal* in Ω, and hence a landing point of at least one equilibrium ray. Every Sturmian measure is a weak* limit of m_{tf_θ} as $t \to \infty$, for some (not necessarily unique) $\theta \in K$. However, in contrast to equilibrium states, Sturmian measures have very weak ergodic properties, in particular zero entropy. The upper semi-continuity of entropy (see Walters [27]) therefore implies that $h(m_{tf_\theta}) \to 0$ as $t \to \infty$, for all $\theta \in K$.

A feature of this example is that the boundary is non-differentiable at a countable dense set of points. The points of non-differentiability are precisely the barycentres of the *atomic* Sturmian measures. Figure 1 shows an approximation to the boundary.

A finite number of extremal points are illustrated, each the barycentre of an atomic Sturmian measure. Although each equilibrium ray lands at a unique point on the boundary, typically many rays can land at the *same* boundary point. This is illustrated in Figure 2, where all rays r_θ are drawn for θ an integer multiple of 0.01.

In particular the boundary points 1, $-1/2$ and $-0.166666 \pm 0.4409585i$ (which are barycentres for atomic measures supported on orbits of period one, two, and three respectively) are seen to be the points at infinity for many rays. It is known that for all $|\theta| \leq 0.14955...$ the ray r_θ lands at $1 \in \mathbb{C}$. For $0.420148 < \theta < 0.579852$ the ray r_θ lands at $-1/2$. For $0.279199 < \theta < 0.367215$ the ray r_θ lands at $-0.166666 + 0.4409585i$ (see Jenkinson [11]). In fact every point of non-differentiability of $\partial\Omega$ is the landing point for many rays. By contrast the points of differentiability of $\partial\Omega$ are landing points for a single ray.

Lebesgue measure is the measure of maximal entropy, and hence its barycentre at 0 is the common basepoint for all rays. It is easy to prove that the barycentre set is symmetric about the real axis (since T commutes with complex conjugation). Indeed the family of equilibrium rays also has real axis symmetry, in the sense that $r_{-\theta}$ is the complex conjugate of r_θ. It follows that the ray r_0 is the real line segment joining 0 to 1, while $r_{1/2}$ is the segment joining 0 to $-1/2$. More generally we observe that all rays r_θ emanate from the origin with angle $2\pi\theta$, a consequence of the following Proposition. An analogous proof works for $z \mapsto z^k$, $k \geq 2$, but the result is not true for general piecewise-onto expanding maps.

Figure 1. Boundary of the barycentre set for the doubling map

Figure 2. Radial foliation by equilibrium rays for the doubling map barycentre set

Proposition 13. *Let $T : S^1 \to S^1$ be the doubling map $z \mapsto z^2$. The induced radial map $\hat{B} : \mathbb{C} \to int(\Omega)$ is holomorphic at the origin, with derivative $\hat{B}'(0) = 1/2$. In particular it is conformal at the origin.*

Proof. Fix $\theta \in K$. We will work with θ-coordinates. We have $r_\theta(t) = b(m_{tf_\theta})$.

The component of this barycentre in the $2\pi\theta$ direction is $I(t,\theta) = \int f_\theta \, dm_{tf_\theta}$, while its component in the (orthogonal) $2\pi\theta - \pi/2$ direction is $J(t,\theta) = \int g_\theta \, dm_{tf_\theta}$, where $g_\theta(x) = \sin 2\pi(x - \theta)$.

Since $\int g_\theta \, dm_{tf_\theta} = \frac{\partial}{\partial s} P(sg_\theta + tf_\theta)|_{s=0}$ (see Ruelle [24], page 133) we have

$$\frac{\partial}{\partial t} J(t,\theta) \Bigg|_{t=t_0} = \frac{\partial^2}{\partial t \partial s} P(sg_\theta + tf_\theta) \Bigg|_{s=0} \Bigg|_{t=t_0} .$$

But Ruelle [24], page 133, gives an explicit formula for the second derivative of pressure, namely

$$\frac{\partial^2}{\partial t \partial s} P(sg_\theta + tf_\theta) \Bigg|_{s=0} \Bigg|_{t=t_0} = \sum_{n=0}^{\infty} \left[\int g_\theta \circ T^n . f_\theta \, dm_{t_0 f_\theta} - \int g_\theta \, dm_{t_0 f_\theta} \int f_\theta \, dm_{t_0 f_\theta} \right].$$

As $t_0 \to 0$ then $m_{t_0 f_\theta}$ converges weak* to the measure of maximal entropy m_0, which is Lebesgue measure. So combining the above we obtain

$$\begin{aligned} \lim_{t_0 \to \infty} \frac{\partial}{\partial t} J(t,\theta) \Bigg|_{t=t_0} &= \sum_{n=0}^{\infty} \left[\int g_\theta(T^n x) . f_\theta(x) \, dx - \int g_\theta(x) \, dx \int f_\theta(x) \, dx \right] \\ &= \sum_{n=0}^{\infty} \int_0^1 \sin 2\pi(2^n x - \theta) \cos 2\pi(x - \theta) \, dx. \end{aligned}$$

Now

$$\begin{aligned} &\int_0^1 \sin 2\pi(2^n x - \theta) \cos 2\pi(x - \theta) \, dx \\ &\qquad = \sin 2\pi\theta \cos 2\pi\theta \left(\int_0^1 \sin 2\pi x \sin(2\pi 2^n x) \, dx - \int_0^1 \cos 2\pi x \cos(2\pi 2^n x) \, dx \right) \\ &\qquad = 0 \qquad \text{for all } n. \end{aligned}$$

Hence

$$\lim_{t_0 \to \infty} \frac{\partial}{\partial t} J(t,\theta) \Bigg|_{t=t_0} = 0.$$

A similar calculation shows that

$$\lim_{t_0 \to \infty} \frac{\partial}{\partial t} I(t,\theta) \Bigg|_{t=t_0} = \sum_{n=0}^{\infty} \int_0^1 \cos 2\pi(2^n x - \theta) \cos 2\pi(x - \theta) \, dx.$$

Here all terms in the sum vanish except for the $n = 0$ term, which equals $1/2$. Thus the derivative in the $2\pi\theta$ direction is precisely $1/2$, independent of θ.

Hence the derivative is $\hat{B}'(0) = 1/2$. $\square$

Remark. We know that $\hat{B}$ is real-analytic as a map $\mathbb{R}^2 \to \mathbb{R}^2$, and holomorphic at 0 as a map $\mathbb{C} \to \mathbb{C}$. However $\hat{B}$ is not globally holomorphic, by Liouville's Theorem, since its image is the bounded set $int(\Omega)$. Moreover, real-analyticity is destroyed at the boundary, since $\partial\Omega$ is not even differentiable.

Section 8. The tent map.

In this section we consider the tent map $T : S^1 \to S^1$ defined by $T(e^{2\pi ix}) = e^{2\pi i\tau(x)}$ where $\tau(x) = 1 - |2x - 1|$. It will be notationally more convenient to talk of periodic orbits for τ rather than for T.

Here the boundary of the barycentre set is somewhat simpler than was the case for the doubling map, though the radial foliation by equilibrium rays is in a sense more complicated. Computational evidence (see Hunt & Ott [10]) suggests that the boundary consists of a countable number of line segments, where the only extremal points are barycentres of ergodic atomic measures supported on the fixed points 0 and 2/3, and the orbits $\mathcal{O}_n$ of the points $1/(2^n + 1)$ for $n \geq 2$. Our own computations support this. The barycentres supported on $\mathcal{O}_n$ converge to the barycentre $1 \in \mathbb{C}$ of the Dirac measure supported on the fixed point 0. In fact (see Figure 3) these barycentres quickly become almost collinear, though their convergence to the point 1 is only of order $1/n$. In Figure 4 we see that (as for the doubling map) many equilibrium rays land at points of non-differentiability of the boundary. Unlike the doubling map, however, these points of non-differentiability are the only extremal points of Ω. The larger the exterior angle of Ω at a particular extremal point w, the larger the θ-interval for which the ray r_θ lands at w. We have used some artistic licence in rendering Figure 4, by selectively choosing values of θ so that the rays are fairly evenly spaced. Had we drawn rays for all θ integer multiples of 0.01, say (as we did for the doubling map in §7), then the rays would appear sparsely distributed in much of $int(\Omega)$ and extremely concentrated near extremal points with large exterior angle.

At certain critical values of θ the set $w(\theta)$ of barycentres with maximal component in the $2\pi\theta$ direction is a line segment. For such critical θ there are two ergodic f_θ-optimal measures, and all convex combinations of these are also f_θ-optimal. For example when $\theta = 5/6$ the two ergodic f_θ-optimal measures are the Dirac measures δ_0, $\delta_{2/3}$ supported on the two fixed points (namely 0 and 2/3 for τ, or 1 and $e^{4\pi i/3}$ for T). We see that $1/2 = f_{5/6}(0) = f_{5/6}(2/3)$ is the optimal ergodic average for $f_{5/6}$ as follows. Note that $f_{5/6}(x) > 1/2$ if and only if $x \in (2/3, 1)$. However if $x \in (2/3, 1)$ then $\tau(x) = 2 - 2x \in (0, 1/3)$. A calculation shows that $f_{5/6}(x) + f_{5/6}(\tau x) < 1$ if $x \in (2/3, 1)$. For any point y, let $R_n \subset \mathbb{N}$ be the set of $i \leq n-1$ for which $\tau^i(y) \in (2/3, 1)$. Let S_n be the set of $i \leq n-1$ such that $i \notin R_n$ and $i - 1 \notin R_n$. Then if $n - 1 \notin R_n$ we have

$$\frac{1}{n} f^n(y) = \frac{1}{n} \left(\sum_{i \in R_n} \left[f_{5/6}(y) + f_{5/6}(\tau y) \right] + \sum_{i \in S_n} f_{5/6}(y) \right) < 1/2.$$

Consequently $\limsup_{n\to\infty} \frac{1}{n} f^n(y) \leq 1/2$, and we are done.

Figure 3. Boundary of the barycentre set for the tent map

Figure 4. Radial foliation by equilibrium rays for the tent map barycentre set

Section 9. Extremal Measures.

Motivated by the behaviour of equilibrium rays, in this section we consider a more general question. For a function $f \in \mathcal{H}(K, \mathbb{R})$, we investigate the nature of the weak* accumulation points, as $t \to \infty$, of the equilibrium states m_{tf}.

Definition 4. *An f-optimal measure m is called f-extremal if it satisfies $h(m) = \sup\{h(\mu) : \mu$ is f-optimal $\}$.*

Lemma 14. *Let $T : K \to K$ be a piecewise-onto expanding map. If $f \in \mathcal{H}(K, \mathbb{R})$ then as $t \to \infty$, any weak* limit point m_∞ of m_{tf} is f-extremal.*

Proof. Suppose there is some f-optimal measure $m \neq m_\infty$ with $h(m) - h(m_\infty) = \epsilon > 0$.

By Lemma 3 (d) we know $h(m_{tf})$ is strictly decreasing with t. The upper semicontinuity of entropy then implies $h(m_\infty) \geq \lim_{t\to\infty} h(m_{tf})$. So $h(m) - \lim_{t\to\infty} h(m_{tf}) \geq \epsilon > 0$.

Choose t_0 sufficiently large so that $h(m_{t_0 f}) - \lim_{t\to\infty} h(m_{tf}) < \epsilon/2$. Thus $h(m) - h(m_{t_0 f}) > \epsilon/2 > 0$. Since m is f-optimal then $\int f\, dm \geq \int f\, dm_{t_0 f}$ and so $t_0 \int f\, dm \geq t_0 \int f\, dm_{t_0 f}$. Combining these inequalities gives

$$h(m) + \int t_0 f\, dm > h(m_{t_0 f}) + \int t_0 f dm_{t_0 f},$$

contradicting the fact that $m_{t_0 f}$ is the equilibrium state for $t_0 f$. $\square$

Coelho [6] made the following three conjectures about Hölder functions f. Firstly, that there are only finitely many ergodic f-extremal measures, $\mu_1, \ldots, \mu_n$ say. Secondly, that the weak* limit $\lim_{t\to\infty} m_{tf}$ always exists, and thirdly, that the limit equals the evenly-weighted centroid $\frac{1}{n}\sum_{i=1}^{n} \mu_i$. Pollicott & Sharp [20] showed the first conjecture to be false. We now show that the third conjecture is false. Namely, we will present two examples where there are only finitely many ergodic extremal measures, where the limit $\lim_{t\to\infty} m_{tf}$ does exist, but yet is not equal to the centroid of the ergodic extremal measures. Coelho's work was in the context of transitive subshifts of finite type, so we present our counterexamples in this context too. They can easily be adapted to the context of piecewise-expanding circle maps.

Let $X = A^{\mathbb{N}}$ be the one-sided full shift on the finite alphabet A, equipped with the product topology. Let $[i_0, \ldots, i_n] = \{x \in X : x_0 = i_0, \ldots, x_n = i_n\}$ denote a cylinder set of length $n + 1$. Let $T : X \to X$ be the shift map given by $(Tx)_n = x_{n+1}$.

Example A. We let $A = \{1, 2, 3\}$ be our alphabet. Let $f : X \to \mathbb{R}$ be the function locally constant on cylinders of length two given by $f[i, j] = M(i, j)$, where

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

There are three ergodic optimal measures for f, each one a Dirac measure supported on a fixed point of T. Let δ_i denote the measure supported on the point $(i, i, i, \ldots)$. Each δ_i has zero entropy, and hence is also f-extremal.

The equilibrium state m_{tf} is a certain measure defined by a column stochastic matrix P_t. That is, the measure m_{tf} on cylinders is given by

$$m_{tf}[i_0, \ldots, i_n] = P_t(i_0, i_1) \ldots P_t(i_{n-1}, i_n) p_t(i_n),$$

where $P_t p_t = p_t$ and $p_t(1)+p_t(2)+p_t(3) = 1$. The formula for P_t (see Parry & Pollicott [18], page 27) is $P_t(i,j) = h_t(i) e^{tf[i,j]} h_t(j)^{-1} \beta_t^{-1}$ where β_t is the maximal positive eigenvalue of the matrix $(e^{tf[i,j]})$, and h_t is the corresponding left eigenvector. An explicit calculation gives

$$P_t = \frac{1}{a(e^t)} \begin{pmatrix} e^{2t} & e^t & b(e^t) \\ e^t & e^{2t} & b(e^t) \\ \frac{1}{b(e^t)} & \frac{1}{b(e^t)} & e^{2t} \end{pmatrix} \quad , \quad p_t = \begin{pmatrix} c(e^t) \\ c(e^t) \\ d(e^t) \end{pmatrix},$$

where

$$a(z) = z^2 + \frac{z}{2} + \frac{1}{2}\sqrt{8+z^2} \quad , \quad b(z) = \frac{2 + z^2 + z\sqrt{8+z^2}}{3z + \sqrt{8+z^2}} \quad ,$$

$$c(z) = \frac{z^3 + z^2\sqrt{8+z^2} + 5z + \sqrt{8+z^2}}{2z^3 + 2z^2\sqrt{8+z^2} + 16z + 4\sqrt{8+z^2}} \quad ,$$

$$d(z) = \frac{6z + 2\sqrt{8+z^2}}{2z^3 + 2z^2\sqrt{8+z^2} + 16z + 4\sqrt{8+z^2}} \quad .$$

Letting $t \to \infty$ we see

$$P_t \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad p_t \to \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}.$$

Thus $\lim_{t\to\infty} m_{tf} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. In particular we do *not* have $\lim_{t\to\infty} m_{tf} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2 + \frac{1}{3}\delta_3$.

Example B. A simple modification of Example A yields a case where there exist positive entropy optimal measures, so that all extremal measures will have positive entropy. We let $A = \{1, \ldots, 6\}$ be a six symbol alphabet. Let $f : X \to \mathbb{R}$ be the function locally constant on cylinders of length two given by $f[i,j] = M(i,j)$, where

$$M = \begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

Here there are many optimal measures. Any measure whose support is contained within either $[1] \cup [2]$, $[3] \cup [4]$, or $[5] \cup [6]$ is f-optimal. There are three ergodic extremal measures, namely $B(1,2)$, $B(3,4)$ and $B(5,6)$, where $B(i,j)$ denotes half-half Bernoulli measure on the symbols i, j.

Analogously to Example A, $\lim_{t\to\infty} m_{tf}$ is $\frac{1}{2}(B(1,2)+B(3,4))$ rather than $\frac{1}{3}(B(1,2)+B(3,4)+B(5,6))$.

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