COMPUTING INVARIANT DENSITIES AND METRIC ENTROPY

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Abstract. We present a method for accurately computing the metric entropy (or, equivalently, the Lyapunov exponent) of the absolutely continuous invariant measure μ for a piecewise analytic expanding Markov map T of the interval. We construct atomic signed measures μ_M supported on periodic orbits up to period M, and prove that $\int \log |T'| \, d\mu_M \to h(\mu)$ super-exponentially fast. We illustrate our method with several examples.

0. The problem

Let $T : I \to I$ be a piecewise C^2 expanding map of the interval. By the Lasota-Yorke theorem [L-Y] we know there exists a T-invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure. If T is topologically mixing then this absolutely continuous invariant measure (abbreviated to a.c.i.m.) is unique and ergodic, with strictly positive C^1 density function $\rho : I \to \mathbb{R}$.

In the special case of the continued fraction transformation $Tx = 1/x \pmod 1$ the density is $\rho(x) = ((1+x)\log 2)^{-1}$. For the Ulam-von Neumann map $Tx = 4x(1-x)$, which is topologically conjugate to the expanding tent map $\tau(x) = 1 - |2x-1|$, the density is known to be $\rho(x) = \pi^{-1}(x(1-x))^{-1/2}$. Finally, Parry [Pa] and Renyi [Renyi] studied the problem for certain piecewise linear examples, including the β-transformation defined by $Tx = \beta x \pmod 1$, for any $\beta > 1$. However, in general there is no explicit formula for the density function, and much recent interest has focused on *approximating* the density [HuntB], [Froy1], [Froy2], [HuntF], [K-M-Y], [M], with particular emphasis on Ulam's method [Ulam]. In this note we will present an alternative approach.

We shall only consider the case where T is a real analytic Markov map. In this case a well-known approach to finding the a.c.i.m. μ is given by the weighted distribution of periodic orbits. More precisely, the sequence of atomic T-invariant probability measures

$$m_M = \frac{\sum_{x \in \mathrm{Fix}(M)} \delta_x / |(T^M)'(x)|}{\sum_{x \in \mathrm{Fix}(M)} 1/|(T^M)'(x)|}, \quad M \geq 1 \tag{0.1}$$

(where the summations are over the set $\mathrm{Fix}(M) = \{x \in I : T^M x = x\}$ of period-M points) converges to μ in the weak-star topology (see [K-H, p. 635]). That is, for each $k \in \mathbb{Z}$,

$$\int_0^1 e^{2\pi i k x} dm_M(x) \to \hat{\mu}(k) := \int_0^1 e^{2\pi i k x} d\mu(x).$$

This convergence can be shown to be exponentially fast (see section 4), the rate being determined by the second eigenvalue of the Perron-Frobenius operator $\mathcal{L}_T$ (defined in section 1).

In this note we will present a rather more efficient method of computing these Fourier coefficients $\hat{\mu}(k)$. We achieve this by a more elaborate regrouping of the periodic points to define new invariant *signed* probability measures μ_M by

$$\mu_M = \frac{1}{N_M} \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\ldots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x\in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1 \\ j\neq i}}^m \sum_{z\in\mathrm{Fix}(n_j)} r(z,n_j) \right) \frac{\delta_x}{|(T^{n_i})'(x)-1|}.$$

where we denote

$$r(x,n) = \frac{1}{n|(T^n)'(x)-1|},$$

and the normalisation constant N_M is simply

$$N_M = \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\ldots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x\in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1 \\ j\neq i}}^m \sum_{z\in\mathrm{Fix}(n_j)} r(z,n_j) \right) \frac{1}{|(T^{n_i})'(x)-1|}.$$

Note in particular that the measures μ_M are supported on those periodic points of period at most M.

The advantage of these more involved definitions is that in the case when T is C^ω we have the following superexponential convergence of the Fourier coefficients.

Theorem 1. *Suppose that $T: I \to I$ is a piecewise C^ω expanding Markov map of the interval, with absolutely continuous invariant probability measure μ. There is a sequence of T-invariant signed probability measures μ_M, supported on the points of period at most M, and $0<\theta<1$, such that for each $k\in\mathbb{Z}$, there exists $C>0$ with*

$$\left| \int_0^1 e^{2\pi k i x} d\mu_M(x) - \hat{\mu}(k) \right| \le C\theta^{M^2}.$$

More generally, for any real analytic function $g: I \to \mathbb{R}$, we can similarly see that $|\int_0^1 g(x)d\mu_M(x) - \int_0^1 g(x)d\mu(x)| \le C\theta^{M^2}$. Of particular interest is the choice $g(x) = \log|T'(x)|$, since by the Rohlin-Pesin equality [Ro], [Pe] we can identify the metric entropy $h_\mu(T)$ with such an integral,

$$h_\mu(T) = \int \log|T'(x)| d\mu(x). \tag{0.2}$$

The ergodic theorem means that $h_\mu(T)$ equals the Lyapunov exponent

$$\lim_{n\to+\infty} \frac{1}{n} \log|(T^n)'(x)|$$

for Lebesgue almost all x.

Combining identity (0.2) with the method of proof of Theorem 1, we also prove

Theorem 2. *Suppose that $T : I \to I$ is a piecewise C^ω expanding Markov map of the interval, with absolutely continuous invariant probability measure μ. There is a sequence of T-invariant signed probability measures μ_M, supported on the points of period at most M, and constants $0 < \theta < 1$, $C > 0$, such that $|h_\mu(T) - \int \log|T'|\, d\mu_M| \le C\theta^{M^2}$.*

We organise the article as follows. In section 1 we collect together a few easy results on expanding piecewise-analytic Markov interval maps, and consider analytically parametrised families of such maps. In section 2 we show that the measures μ_M converge to the a.c.i.m. μ. In section 3 we consider the speed of convergence, and prove our theorems. In section 4 we use our methods to estimate the entropy of the a.c.i.m. for two families of examples.

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1. Some simple properties of the density and metric entropy

Consider a piecewise C^ω expanding Markov interval map $T : I \to I$. By this we mean

(a) there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ such that T is real-analytic on each $[a_i, a_{i+1}]$ (i.e. extends to a holomorphic map on some complex neighbourhood of $[a_i, a_{i+1}]$),

(b) there exists $\kappa > 1$ such that $|T'(x)| \ge \kappa$ for all $x \in I$,

We shall also assume for convenience that $T[a_i, a_{i+1}] = I$ for all i. This condition is not essential, but eases the exposition in the sequel. In particular it implies T is topologically mixing. A more general definition of a Markov map would allow each image $T[a_i, a_{i+1}]$ to be some union $\cup_{j\in J}[a_j, a_{j+1}]$ of pieces in the partition. Under this hypothesis, the invariant density ρ in Lemma 1 would only be C^ω on each piece of the partition. Moreover, the various function spaces we consider would be defined relative to the partition, and would consist of functions with some prescribed regularity on each piece of the partition.

Let μ denote the unique ergodic a.c.i.m. [C-E] and let $\rho \in L^1(I)$ denote the associated density (or Radon-Nikodym derivative) i.e., $d\mu(x) = \rho(x)d\lambda(x)$, where λ is Lebesgue measure. In general it is not possible to find an explicit expression for the density ρ.

We can define the Perron-Frobenius operator $\mathcal{L}_T : C^0(I) \to C^0(I)$ by

$$\mathcal{L}_T k(x) = \sum_{Ty=x} \frac{k(y)}{|T'(y)|}, \quad k \in C^0(I).$$

Recall that the space of real-analytic functions $C^\omega(I)$ is a Fréchet space. If we fix some open complex neighbourhood U of I, then the space of bounded holomorphic functions on U is a Banach space with respect to the uniform norm.

Lemma 1. *The density ρ is a real-analytic function, and satisifies $\mathcal{L}\rho = \rho$.*

This is a standard result following from the change of variable formula (cf. [K-H, p. 186]), the latter observation following immediately from the fact that $\mathcal{L}$ preserves $C^\omega(I)$, i.e., $\mathcal{L}C^\omega(I) \subset C^\omega(I)$.

Lemma 2. *Suppose the metric entropy $h_\mu(T)$ is equal to the topological entropy $h(T)$. Then the map T is C^ω conjugate to a piecewise linear map whose slope on each partition piece is $\pm p$, where p is the number of pieces in the Markov partition.*

A simple calculation with Radon-Nikodym derivatives [K-H, p. 187] gives

$$\frac{d\mu T}{d\mu}(x) = \frac{d(\mu T)}{d(\lambda T)}\frac{d(\lambda T)}{d\lambda}\frac{d\lambda}{d\mu} = \rho(Tx)|T'(x)|\rho(x)^{-1}. \tag{1.1}$$

By T-invariance of μ the jacobian $d\mu T/d\mu$ must satisfy

$$\sum_{Ty=x}\left(\frac{d\mu T}{d\mu}(y)\right)^{-1} = 1, \text{ a.e.} \tag{1.2}$$

If $h_\mu(T) = h(T)$, then μ is the unique measure of maximal entropy, so (see [Wal1]) $-\log|T'|$ is an essential coboundary. Thus $-\log\frac{d\mu T}{d\mu}$ is also an essential coboundary (i.e. $-\log\frac{d\mu T}{d\mu} = vT - v + c$ for some real c and bounded measurable function v). But then (1.2) implies $\frac{d\mu T}{d\mu}$ must itself be constant, and that this constant is precisely p.

Therefore, if we assume that 0 is a fixed point, for convenience, then by (1.1) we deduce T is conjugate to the required piecewise linear map by the conjugacy $h(x) = \int_0^x \rho(x)dx$.

Lemma 3. *Let T_ε $(-a < \varepsilon < a)$ be a family of C^ω expanding Markov maps, which vary analytically with the parameter ε, and with a.c.i.m. μ_ε. Then the associated metric entropies $h(\mu_\varepsilon)$ also vary analytically.*

The invariant density ρ_ε for T_ε satisfies $\mathcal{L}_{T_\varepsilon}\rho_\varepsilon = \rho_\varepsilon$ (by Lemma 1). Moreover, the maximal eigenvalue 1 for $\mathcal{L}_{T_\varepsilon}$ is simple and isolated. It follows from standard analytic perturbation theory that the map $\varepsilon \mapsto \rho_\varepsilon \in C^\omega(I)$ is C^ω [Kato]. By (0.2) we can write $h(\mu_\varepsilon) = \int \log|T_\varepsilon'|\rho_\varepsilon(x)dx$, from which Lemma 3 immediately follows.

Let us now consider the specific family of maps $T_\varepsilon : [0,1] \to [0,1]$ defined by

$$T_\varepsilon(x) = 2x + \varepsilon\sin 2\pi x \pmod 1, \text{ for } -\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}.$$

These maps are all analytic and expanding. They are also topologically conjugate to the standard doubling map $T_0 : I \to I$, from which we deduce that the topological entropy $h(T_\varepsilon)$ is equal to $\log 2$ (indeed the same is true for any degree-2 expanding map [K-H, Theorem 2.4.2]).

By the variational principle (see for example [Wal2]), the metric entropy of the a.c.i.m. μ_ε must necessarily satisfy $h(\mu_\varepsilon) \le \log 2$. The following lemma is an easy consequence of Lemma 2.

Lemma 4. *For $\varepsilon \ne 0$ the map T_ε has metric entropy $h(\mu_\varepsilon) < \log 2$*

If we assume for a contradiction that $h(\mu_\varepsilon) = \log 2$ then Lemma 1 implies that T_ε is C^ω conjugate to the linear map with derivative 2. However, when we compute the derivative at the fixed point 0 we see $(T_\varepsilon')(0) = 2(1-\varepsilon)$, whereas it is required to take the value $T_\varepsilon'(0) = 2$, giving the required contradiction.

Indeed we can derive a second order approximation of $h(\mu_\varepsilon)$ as follows.

Proposition 1. *$h(\mu_\varepsilon) = \log 2 - \frac{\pi^2}{4}\varepsilon^2 + O(\varepsilon^3)$ as $\varepsilon \to 0$.*

Proof. The Rohlin-Pesin identity gives

$$h(\mu_\varepsilon) = \int \log |T'_\varepsilon(x)| \rho_\varepsilon(x) dx, \tag{1.3}$$

where ρ_ε is the invariant density for T_ε. By Lemma 3, we know the entropy $h(\mu_\varepsilon)$ varies analytically, so let us write $h(\mu_\varepsilon) = \log 2 + h_1\varepsilon + h_2\varepsilon^2 + O(\varepsilon^3)$, where the coefficients h_1, h_2 are to be determined.

We have the expansions

$$\log |T'_\varepsilon(x)| = \log 2 + \varepsilon\pi\cos(2\pi x) - \frac{\varepsilon^2\pi^2}{2}\cos^2(2\pi x) + O(\varepsilon^3)$$

and

$$\rho_\varepsilon(x) = 1 + \rho^1(x)\varepsilon + \rho^2(x)\varepsilon^2 + O(\varepsilon^3).$$

The functions ρ^1, ρ^2 satisfy $\int \rho^1(x)\,dx = 0 = \int \rho^2(x)\,dx$, since $\int \rho_\varepsilon(x)\,dx = 1$ for all ε. Substituting the expansions into (1.3), and comparing coefficients, we immediately deduce that $h_1 = 0$. Comparing second order terms we have

$$h_2 = \log 2 \int \rho^2(x) dx + \pi \int \cos(2\pi x)\rho^1(x) dx - \frac{\pi^2}{2}\int \cos^2(2\pi x) dx. \tag{1.4}$$

The first term on the right-hand side of (1.4) is zero, by the previous comment.

We now claim that the function ρ^1 is identically zero. Once we have established this, the second term on the right-hand side of (1.4) will vanish to leave $h_2 = -\frac{\pi^2}{2}\int \cos^2(2\pi x)dx = -\pi^2/4$, as required.

Let σ_ε be the inverse branch of T_ε which is close to $x \mapsto x/2$. We can write $\sigma_\varepsilon(x) = x/2 + \varepsilon\sigma^1(x) + O(\varepsilon^2)$, and using the identity $x = T_\varepsilon(\sigma_\varepsilon(x))$, we derive $\sigma^1(x) = -\frac{1}{2}\sin(\pi x)$. If τ_ε is the inverse branch near $x \mapsto (x+1)/2$, then we can write $\tau_\varepsilon(x) = (x+1)/2 + \varepsilon\tau^1(x) + O(\varepsilon^2)$, and a similar argument gives $\tau^1(x) = \frac{1}{2}\sin(\pi x) = -\sigma^1(x)$.

Since ρ_ε is a fixed point of the Perron-Frobenius operator $\mathcal{L}_{T_\varepsilon}$ we have

$$\rho_\varepsilon(x) = \mathcal{L}_{T_\varepsilon}\rho_\varepsilon(x) = \rho_\varepsilon(\sigma_\varepsilon x)\sigma'_\varepsilon(x) + \rho_\varepsilon(\tau_\varepsilon x)\tau'_\varepsilon(x). \tag{1.5}$$

Substituting the various series expansions into (1.5), and considering the linear term in ε, we obtain

$$\begin{aligned}\rho^1(x) &= \frac{1}{2}\left(\rho^1\left(\frac{x}{2}\right) + \rho^1\left(\frac{x+1}{2}\right)\right) + (\sigma^1)'(x) + (\tau^1)'(x)\\ &= \frac{1}{2}\left(\rho^1\left(\frac{x}{2}\right) + \rho^1\left(\frac{x+1}{2}\right)\right).\end{aligned}$$

We thus deduce that $\rho^1 = \mathcal{L}_{T_0}\rho^1$, so that ρ^1 lies in the eigenspace spanned by the invariant density for the doubling map T_0. That is, ρ^1 is a constant. Having already observed that $\int \rho^1(x)dx = 0$, we finally deduce that ρ^1 is identically zero, as required.

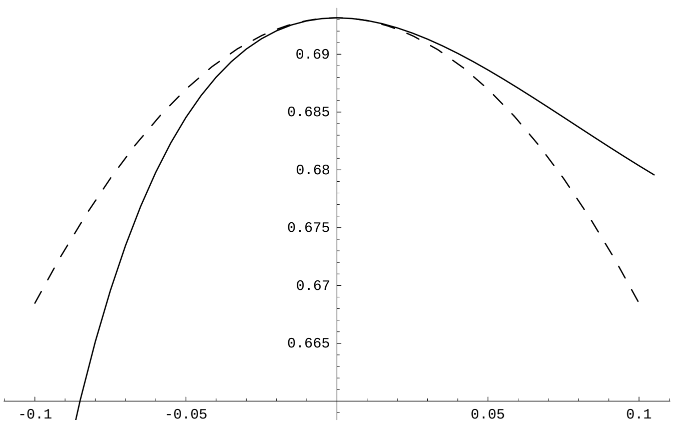

Figure 1. Entropy $h(\mu_\varepsilon)$ and second order approximation $\log 2 - \frac{\pi^2}{4}\epsilon^2$ (dashed)

In Figure 1 we compare $h(\mu_\varepsilon)$ to the second order approximation derived in Proposition 1.

2. Convergence of the measures μ_M

In this section we shall show that the measures μ_M converge to the a.c.i.m. μ. In section 3 we shall study the rate of convergence.

In section 1, we considered the Perron-Frobenius operator. In our proof of Theorem 1 we need to modify this definition slightly. More generally, for any Hölder continuous function $f : I \to \mathbb{R}$ we can define the *transfer operator* $\mathcal{L}_f : C^0(I) \to C^0(I)$ by

$$\mathcal{L}_f k(x) = \sum_{Ty=x} k(y) e^{f(y)}.$$

This operator $\mathcal{L}_f$ has a simple isolated maximal eigenvalue at the point $e^{P(f)}$, where

$$P(f) = P(f,T) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}(n)} e^{f^n(x)}$$

is the *pressure* of f, and we denote $f^n(x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$[Bowen].

Given an analytic function $g : I \to \mathbb{R}$ and $t \in \mathbb{R}$ we choose

$$f(x) = -\log |T'(x)| + tg(x).$$

The maximal eigenvalue of the corresponding transfer operator $\mathcal{L}_{-\log|T'|+tg}$ is therefore $\exp P(-\log|T'| + tg)$. We define $z(t)$ to be its reciprocal

$$z(t) := \exp\left[-P(-\log|T'| + tg)\right].$$

Note that setting $t = 0$ gives $z(0) = \exp\left[-P(-\log|T'|)\right] = 1$, since 1 is the maximal eigenvalue of the Perron-Frobenius operator $\mathcal{L}_T$. We want to relate this to the integral of the function g with respect to the a.c.i.m. μ via the following formula.

Lemma 5.
$$\int g\, d\mu = \frac{d}{dt} P(-\log|T'| + tg)\bigg|_{t=0} .$$

(see [Ru2, p.133], [Pa-Pol, p.60]*).*

In particular, since $z(0) = 1$, Lemma 5 implies that

$$\int g\, d\mu = -\frac{d}{dt}\log z(t)\bigg|_{t=0} = -\frac{z'(0)}{z(0)} = -z'(0). \tag{2.1}$$

We next want to relate the integral to the periodic points of $T : I \to I$. For a point $x \in \mathrm{Fix}(n)$ we denote

$$r(x, n) = \frac{1}{n|(T^n)'(x) - 1|}.$$

Summing over all such points we let

$$a_n = a_n(t) = \sum_{x \in \mathrm{Fix}(n)} r(x, n) e^{tg^n(x)},$$

and then formally define the *Fredholm determinant* $F(z, t)$ by

$$F(z,t) = \exp\left(-\sum_{n=1}^{\infty} a_n(t) z^n\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in\mathrm{Fix}(n)} \frac{e^{tg^n(x)}}{|(T^n)'(x) - 1|}\right).$$

In section 3 we will show that in fact $F(z, t)$ is an entire function of both z and t. For the moment let us assume this, and relate F to the expression (2.1) for $\int g\, d\mu$. The first step is the following lemma.

Lemma 6. *The function $z(t) := \exp\left[-P(-\log|T'| + tg)\right]$ gives an implicit solution to $F(z(t), t) = 0$ satisfying $z(0) = 1$.*

Proof. Note that $z(t) = \exp\left[-P(-\log|T'| + tg)\right] = \limsup_{n\to\infty} |a_n(t)|^{1/n}$ is the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n(t) z^n$, and hence a zero of F

(cf. [Ru2, p. 142], where similar statements hold for zeta functions). The fact that $z(0) = 1$ was noted above.

Now by the implicit function theorem we have that

$$z'(0) = -\frac{D_2F(z(0),0)}{D_1F(z(0),0)} = -\frac{D_2F(1,0)}{D_1F(1,0)} \tag{2.2}$$

where $D_iF(z,t)$, $i = 1,2$, denotes the partial derivative of F with respect to the i^{th} variable. Thus by (2.1) we can write

$$\int g\,d\mu = z'(0) = \frac{D_2F(1,0)}{D_1F(1,0)}. \tag{2.3}$$

We will now use the expression (2.3) to compute $\int g\,d\mu$. Both numerator and denominator can be expressed as series of rapidly decreasing terms, as follows. Using the expansion $e^{-x} = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m!}$ we can expand

$$F(z,t) = 1 + \sum_{N=1}^{\infty} C_N(t) z^N,$$

where

$$C_N(t) = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\ldots+n_m=N}} \frac{(-1)^m}{m!} a_{n_1}(t)\dots a_{n_m}(t). \tag{2.4}$$

(Note the above summation is over all m-tuples of positive integers whose sum is equal to N). With this expansion for $F(z,t)$ we can now calculate its partial derivatives, as follows.

$$D_1F(z,t) = \sum_{N=1}^{\infty} NC_N(t)z^{N-1}$$

$$D_2F(z,t) = \sum_{N=1}^{\infty} C_N'(t)z^{N}$$

We can explicitly compute the derivative $C_N'(t)$ as

$$\begin{aligned} C_N'(t) &= \frac{d}{dt}\left(\sum_{\substack{(n_1,\dots,n_m)\\ n_1+\ldots+n_m=N}} \frac{(-1)^m}{m!} a_{n_1}(t)\dots a_{n_m}(t)\right) \\ &= \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\ldots+n_m=N}} \frac{(-1)^m}{m!} \sum_{i=1}^{m} a_{n_i}'(t) \prod_{\substack{j=1\\ j\neq i}}^{m} a_{n_j}(t). \end{aligned}$$

Setting $t = 0$, and recalling the notation $g^{n_i}(x) = g(x)+g(Tx)+\ldots+g(T^{n_i-1}x)$, we obtain the following expression:

$$C'_N(0) = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x\in \mathrm{Fix}(n_i)} r(x,n_i) g^{n_i}(x) \prod_{\substack{j=1\\ j\neq i}}^m \sum_{z\in \mathrm{Fix}(n_j)} r(z,n_j).$$

Now

$$\sum_{x\in \mathrm{Fix}(n_i)} r(x,n_i) g^{n_i}(x) = \sum_{x\in \mathrm{Fix}(n_i)} \frac{1}{|(T^{n_i})'(x)-1|}\frac{g^{n_i}(x)}{n_i} = \sum_{x\in \mathrm{Fix}(n_i)} \frac{g(x)}{|(T^{n_i})'(x)-1|},$$

so that

$$\begin{aligned} C'_N(0) &= \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x\in \mathrm{Fix}(n_i)} \frac{g(x)}{|(T^{n_i})'(x)-1|} \prod_{\substack{j=1\\ j\neq i}}^m \sum_{z\in \mathrm{Fix}(n_j)} r(z,n_j) \\ &= \int g \, d\nu_N, \end{aligned}$$

where we define

$$\nu_N = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x\in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\neq i}}^m \sum_{z\in \mathrm{Fix}(n_j)} r(z,n_j) \right) \frac{\delta_x}{|(T^{n_i})'(x)-1|}. \tag{2.5}$$

The sum $\nu_1 + \ldots + \nu_M$ is a finite signed measure, whose support is a union of periodic orbits. Since all points in a given orbit carry equal mass then $\nu_1+\ldots+\nu_M$ is T-invariant. Suitably normalized it gives a signed T-invariant probability measure μ_M. This normalization is obtained by summing the first M coefficients in the power series expansion for $D_1F(z,0)$. Thus we define

$$\mu_M = \frac{\sum_{N=1}^M \nu_N}{\sum_{N=1}^M N C_N(0)}. \tag{2.6}$$

Proposition 2. *The measures μ_M converge to the a.c.i.m. μ in the weak* topology.*

Proof. Let $g(x) = e^{2\pi i k x}$. Then by (2.3), (2.5), (2.6) we see $\int e^{2\pi i k x} d\mu_M \to \hat{\mu}(k)$ as $M \to \infty$.

Remark. In general the signed measures μ_M need not be positive measures. To see this it suffices to find some M such that $\nu_1 + \ldots + \nu_M$ gives mass of opposite sign to two of its atoms.

Suppose, for example, that T is symbolically the full shift on two symbols, with two fixed points x_0 and x_1, and a single period-two orbit $\{x_{01}, x_{10}\}$. Let $e_i = |T'(x_i) - 1|^{-1}$ for $i = 1, 2$.

Using (2.5) for $N = 1$ and $N = 2$ gives $(\nu_1 + \nu_2)(\{x_0\}) = e_0\,[(e_0 + e_1) - 1]$ and $(\nu_1 + \nu_2)(\{x_1\}) = e_1\,[(e_0 + e_1) - 1]$. These weights are both of the same sign, but this sign might be either positive or negative depending on the derivative of T at x_0 and x_1. In particular we can find T with $T'(x_0) = \frac{3}{2}$ and $T'(x_1) = 3$, say, in which case $e_0 + e_1 = \frac{5}{2} > 1$ so that $(\nu_1 + \nu_2)(\{x_0\}) > 0$ and $(\nu_1 + \nu_2)(\{x_1\}) > 0$.

However the period-two point x_{01} is given zero weight by ν_1, and negative weight by ν_2, so that $(\nu_1 + \nu_2)(\{x_{01}\}) < 0$ (the same is also true for x_{10}).

3. Speed of approximation

So far in deriving our formulae, we have assumed that $F(z,t)$ is an entire function of both z and t. We now justify this. First recall (see [Gr]) that a bounded linear operator $L : \mathcal{B} \to \mathcal{B}$ of a Banach space $\mathcal{B}$ over $\mathbb{C}$ is called *nuclear* if there exist normalised $u_i \in B$, normalised $l_i \in B^*$, and $\lambda_i \in \mathbb{C}$ with $\sum_i |\lambda_i| < \infty$ such that

$$Lw = \sum_{i=1}^{\infty} \lambda_i l_i(w) u_i.$$

Lemma 7. *$F(z,t)$ is an entire function of both z and t.*

Proof. For each piece $[a_i, a_{i+1}]$ of the Markov partition we let $T_i : I \to [a_i, a_{i+1}]$ denote the corresponding inverse branch of T (i.e. $T_i \circ T = id$ on $[a_i, a_{i+1}]$), and $z_i \in [a_i, a_{i+1}]$ the fixed point of T_i.

For $\underline{i} = (i_1, \ldots, i_n)$, let $T_{\underline{i}}$ denote the composition $T_{i_1} \circ \ldots \circ T_{i_n}$, and write $|\underline{i}| = n$. For each such $\underline{i}$, let $z_{\underline{i}} \in I$ denote the unique fixed point of $T_{\underline{i}}$, and let $I_{\underline{i}}$ denote the image $T_{\underline{i}} I$.

Let us choose n sufficiently large so that, for each $\underline{i}$ with $|\underline{i}| = n$, there is an open complex disc $D_{\underline{i}} \supset I_{\underline{i}}$ of radius $d_{\underline{i}}$, centered at $z_{\underline{i}}$, such that

(a) for each $1 \le j \le p$, T_j extends holomorphically to the union $D^{(n)} := \cup_{|\underline{i}|=n} D_{\underline{i}}$, with $\overline{T_j(D^{(n)})}$ a proper subset of D_j,

(b) $e^{tg}/|T'|$ extends to a holomorphic function on each $D_{\underline{i}}$, and hence to a holomorphic function (which we denote by ϕ) on the disjoint union $D_n := \coprod_{|\underline{i}|=n} D_{\underline{i}}$.

Let $\mathcal{B}$ denote the space of functions which are holomorphic on the disjoint union D_n and continuous on $\overline{D}_n$. Equipped with the uniform norm, $\mathcal{B}$ is a Banach space. The operator $\mathcal{L}_{-\log|T'|+tg}$ is well-defined on $\mathcal{B}$, and takes the form

$$\mathcal{L}_{-\log|T'|+tg} w(z) = \sum_{j=1}^{p} \phi(T_j(z)) w(T_j(z)).$$

This operator is then nuclear (see [Ru1, p. 236]), and in particular the trace $\mathrm{trace}(\mathcal{L}^m_{-\log|T'|+tg}) = \sum_{i=1}^{\infty} e_i^m < \infty$ is well-defined for each m, where the e_i are the eigenvalues of $\mathcal{L}_{-\log|T'|+tg}$ counted with multiplicity. By Fredholm theory [Gr] we have the trace formulae

$$\mathrm{trace}(\mathcal{L}^m_{-\log|T'|+tg}) = \sum_{x \in \mathrm{Fix}(m)} \frac{e^{tg^m(x)}}{|1-(T^m)'(x)|},$$

and we can identify

$$F(z,t) = \det(I - z\mathcal{L}_{-\log|T'|+tg}) = \exp\left(-\sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{trace}(\mathcal{L}^m_{-\log|T'|+tg})\right)$$

and this is an entire function of both z and t [Ru1].

The above lemma immediately implies the superexponential decay of the power series coefficients $C_N(t)$ of $F(z,t)$. We now give a more precise bound on the rate of decay.

Lemma 8. *There exists $0<\theta<1$ such that, for any $t\in\mathbb{C}$, $C_N(t)=O(\theta^{N^2})$.*

Proof. To ease the notation, let us write $\mathcal{L}=\mathcal{L}_{-\log|T'|+tg}$, so that

$$\mathcal{L}w(z)=\sum_{j=1}^{p}\phi(T_jz)w(T_jz)=\sum_{j=1}^{p}\mathcal{L}_jw(z),$$

where $\mathcal{L}_jw(z):=\phi(T_jz)w(T_jz)$. Suppose n is chosen sufficiently large as in Lemma 7, and that our Banach space $\mathcal{B}$ is as before. Let $\underline{i}=i_1\dots i_n$, and let $\underline{i}'=i_1\dots i_{n-1}$ be its length-$(n-1)$ prefix.

We write $\phi_{\underline{i}}:=\phi|_{D_{\underline{i}}}$. Observe that, for each j, the function $\mathcal{L}_jw$ is defined by

$$(\mathcal{L}_jw)_{\underline{i}}=w_{\underline{j}}\circ T_j.\phi_{\underline{j}}\circ T_j, \tag{3.1}$$

where $\underline{j}=j\underline{i}'$.

Define $D_{\underline{i}'}$ to be the disjoint union $\coprod_{k=1}^{p}D_{\underline{i}'k}$, and note that

$$\overline{T_j(D_{\underline{i}'})}\subset D_{\underline{j}}.$$

Let $B_{\underline{j}}$ be the smallest disc, centered at $z_{\underline{j}}$, which contains $\overline{T_j(D_{\underline{i}'})}$. Let $\beta_{\underline{j}}=\max_{x\in\overline{T_j(D_{\underline{i}'})}}|z_{\underline{j}}-x|$ denote the radius of $B_{\underline{j}}$. Clearly we have $\beta_{\underline{j}}<d_{\underline{j}}$, so define the *contraction ratio* $\alpha_{\underline{j}}=\beta_{\underline{j}}/d_{\underline{j}}<1$.

Let $\Gamma_{\underline{j}}$ be a circle of radius $\gamma_{\underline{j}}$ centred at $z_{\underline{j}}$, and such that $\beta_{\underline{j}}<\gamma_{\underline{j}}<d_{\underline{j}}$. Clearly for any given $\epsilon>0$ we may choose $\Gamma_{\underline{j}}$ such that $\beta_{\underline{j}}/\gamma_{\underline{j}}<(1+\epsilon)\alpha_{\underline{j}}$.

Cauchy's formula means that, for $z\in D_{\underline{i}}\subset D_{\underline{i}'}$, we can express

$$(\mathcal{L}_jw)_{\underline{i}}(z)=w_{\underline{j}}(T_jz)\phi_{\underline{j}}(T_jz)=\frac{1}{2\pi i}\int_{\Gamma_{\underline{j}}}\frac{w_{\underline{j}}(\xi)\phi_{\underline{j}}(\xi)}{\xi-T_jz}d\xi.$$

For $z\in D_{\underline{i}}$, $\xi\in\Gamma_{\underline{j}}$ we write

$$\begin{aligned}(\xi-T_jz)^{-1}&=\left(\xi-z_{\underline{j}}\right)^{-1}\left(1-\frac{T_jz-z_{\underline{j}}}{\xi-z_{\underline{j}}}\right)^{-1}\\&=\left(\xi-z_{\underline{j}}\right)^{-1}\sum_{r=0}^{\infty}\left(\frac{T_jz-z_{\underline{j}}}{\xi-z_{\underline{j}}}\right)^{r}.\end{aligned}$$

Note this expansion is valid because $|T_jz-z_{\underline{j}}|<|\xi-z_{\underline{j}}|$, since the circle $\Gamma_{\underline{j}}$ is outside the image $T_j(D_{\underline{i}'})$. Therefore, for $z\in D_{\underline{i}}\subset D_{\underline{i}'}$ we have

$$(\mathcal{L}_jw)_{\underline{i}}(z)=\sum_{r=0}^{\infty}\left(T_jz-z_{\underline{j}}\right)^{r}\frac{1}{2\pi i}\int_{\Gamma_{\underline{j}}}\frac{w_{\underline{j}}(\xi)\phi_{\underline{j}}(\xi)}{\left(\xi-z_{\underline{j}}\right)^{r+1}}d\xi,$$

so that

$$(\mathcal{L}w)_{\underline{i}}(z)=\sum_{r=0}^{\infty}\sum_{j=1}^{p}\left(T_jz-z_{\underline{j}}\right)^{r}\frac{1}{2\pi i}\int_{\Gamma_{\underline{j}}}\frac{w_{\underline{j}}(\xi)\phi_{\underline{j}}(\xi)}{\left(\xi-z_{\underline{j}}\right)^{r+1}}d\xi. \tag{3.2}$$

Now define $v_{\underline{j},r} \in \mathcal{B}$ and $m_{\underline{j},r} \in \mathcal{B}^*$ by

$$v_{\underline{j},r}(z) := \begin{cases} \left(T_{\underline{j}} z - z_{\underline{j}}\right)^r & \text{if } z \in D_{\underline{i}'} \\ 0 & \text{if } z \notin D_{\underline{i}'}. \end{cases}$$

and

$$m_{\underline{j},r}(w) := \frac{1}{2\pi i} \int_{\Gamma_{\underline{j}}} \frac{w_{\underline{j}}(\xi)\phi_{\underline{j}}(\xi)}{\left(\xi - z_{\underline{j}}\right)^{r+1}} d\xi.$$

From (3.2) we then have

$$\mathcal{L}w = \sum_{r=0}^{\infty} \sum_{|\underline{j}|=n} m_{\underline{j},r}(w) v_{\underline{j},r}.$$

Now we just normalise by setting

$$u_{\underline{j},r} = \frac{v_{\underline{j},r}}{||v_{\underline{j},r}||_\infty} = \frac{v_{\underline{j},r}}{\beta_{\underline{j}}^r} \quad \text{and} \quad l_{\underline{j},r} = \frac{m_{\underline{j},r}}{||m_{\underline{j},r}||_\infty},$$

so that

$$\mathcal{L}w = \sum_{r=0}^{\infty} \sum_{|\underline{j}|=n} \lambda_{\underline{j},r} l_{\underline{j},r}(w) u_{\underline{j},r}, \tag{3.3}$$

for coefficients $\lambda_{\underline{j},r} = ||v_{\underline{j},r}||_\infty ||m_{\underline{j},r}||_\infty$. To estimate the decay of these coefficients as $r \to \infty$, we have

$$\lambda_{\underline{j},r} = ||v_{\underline{j},r}||_\infty ||m_{\underline{j},r}||_\infty \leq \beta_{\underline{j}}^r \frac{||\phi||_\infty}{2\pi\gamma_{\underline{j}}^{r+1}} < \frac{||\phi||_\infty}{2\pi\gamma_{\underline{j}}}(1+\epsilon)\alpha_{\underline{j}}^r.$$

Taking $\epsilon > 0$ sufficiently small, we may choose $\max_{|\underline{j}|=n} \alpha_{\underline{j}} < \alpha < 1$ such that

$$\lambda_{\underline{j},r} \leq K\alpha^r \tag{3.4}$$

for all $|\underline{j}| = n$, where

$$K = ||\phi||_\infty \left(2\pi \min_{|\underline{j}|=n} \gamma_{\underline{j}}\right)^{-1} = Q||e^{tg}||_\infty, \tag{3.5}$$

and Q is a constant depending only on T.

Re-labelling the various functions, functionals, and coefficients in (3.3) we write

$$\mathcal{L}w = \sum_{q=0}^{\infty} \lambda_q l_q(w) u_q,$$

where the λ_q are non-increasing.

For any fixed r, there were p^n functions of the form $u_{\underline{j},r}$, and p^n functionals of the form $l_{\underline{j},r}$, so choosing $\theta_0 = \alpha^{1/p^n} < 1$, the estimate (3.4) implies

$$\lambda_q \leq K\theta_0^q. \tag{3.6}$$

We now estimate the coefficients $C_N(t)$ in the power series expansion of the Fredholm determinant $F(z,t)$. Formally

$$\det(I - z\mathcal{L}) = \begin{vmatrix} 1 - z\lambda_1 l_1(u_1) & -z\lambda_1 l_1(u_2) & -z\lambda_1 l_1(u_3) & \dots \\ -z\lambda_2 l_2(u_1) & 1 - z\lambda_2 l_2(u_2) & -z\lambda_2 l_2(u_3) & \dots \\ -z\lambda_3 l_3(u_1) & -z\lambda_3 l_3(u_2) & 1 - z\lambda_3 l_3(u_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

so we can express

$$C_N(t) = \sum_{i_1 < \ldots < i_N} \lambda_{i_1} \ldots \lambda_{i_N} \mathrm{Det}\left[\left(l_{i_\alpha}(u_{i_\beta}) \right)_{1 \leq \alpha, \beta \leq N} \right]$$

(cf. [Gr, p. 17] for details). Now since the l_{i_α} and u_{i_β} are both normalised, we have $|l_{i_\alpha}(u_{i_\beta})| \leq 1$ for all $1 \leq \alpha, \beta \leq N$. Thus $\mathrm{Det}\left(l_{i_\alpha}(u_{i_\beta}) \right)_{1 \leq \alpha, \beta \leq N} \leq N^{N/2}$.

It remains to estimate the term $\sum_{i_1 < \ldots < i_N} \lambda_{i_1} \ldots \lambda_{i_N}$, which we note is just the N^{th} term in the series expansion of the Euler product $\prod_{q=1}^{\infty}(1 + \lambda_q z)$. Following [Fried, p. 507], the identity

$$\sum_{i_1 < \ldots < i_N} \theta_0^{i_1 + \ldots + i_N} = \frac{\theta_0^{N(N+1)/2}}{(1 - \theta_0) \ldots (1 - \theta_0^N)},$$

together with our estimate (3.6) on the decay rate of the λ_q gives

$$\sum_{i_1 < \ldots < i_N} \lambda_{i_1} \ldots \lambda_{i_N} \leq \frac{K^N \theta_0^{N(N+1)/2}}{(1 - \theta_0) \ldots (1 - \theta_0^N)},$$

so that

$$|C_N(t)| \leq \frac{N^{N/2} K^N \theta_0^{N(N+1)/2}}{(1 - \theta_0) \ldots (1 - \theta_0^N)} \leq A N^{N/2} K^N \theta_0^{N(N+1)/2}, \tag{3.7}$$

where $A = \prod_{j=1}^{\infty}(1 - \theta_0^j)^{-1} < \infty$. Setting $\theta = \theta_0^{1/2}$, we have $C_N(t) = O(\theta^{N^2})$, which concludes the lemma.

Remark. *The rate θ is independent of t and g. The constant K does depend on t and g, but note that setting $t = 0$ removes the dependence on g (see (3.5)). Thus $C_N(0)$ is $O(\theta^{N^2})$, where the implicit constant is independent of the function g.*

The next lemma extends our estimates on $C_N(t)$ to the coefficients $NC_N(0)$ and $C_N'(0)$ of the power series expansions for the partial derivatives of F.

Lemma 9. *We can bound* $|NC_N(0)| = O(\theta^{N^2})$ *and* $|C_N'(0)| = O(\theta^{N^2})$.

Proof. Clearly the bound on $NC_N(0)$ follows immediately from that on $C_N(0)$. Moreover, by the above remark, the implicit constant is independent of the function g.

To estimate $|C_N'(0)|$ we first combine (3.5) and (3.7) to obtain

$$|C_N(t)| \le AN^{N/2}Q^N\theta^{N(N+1)}||e^{tg}||_\infty^N.$$

Then by Cauchy's theorem we integrate around the unit circle, say, to obtain

$$|C_N'(0)| = \left|\frac{1}{2\pi i}\int_{|t|=1}\frac{C_N(t)}{t^2}dt\right| \le \frac{A}{2\pi}N^{N/2}Q^N\theta^{N(N+1)}e^{N||g||_\infty}, \tag{3.8}$$

and the result follows. Note however that $C_N'(0)$ does depend on g.

Now we can finally complete the proof of Theorems 1 and 2.

Setting $g(x) = e^{2\pi ikx}$, we use Lemmas 8 and 9 to deduce that there exists $0 < \theta < 1$ such that

$$\int g\,d\mu = \frac{D_2F(1,0)}{D_1F(1,0)} = \frac{\sum_{N=1}^M C_N'(0)}{\sum_{N=1}^M NC_N(0)} + O(\theta^{M^2}) = \int g\,d\mu_M + O(\theta^{M^2}).$$

The proof of Theorem 2 follows in the same way, though here we make the choice $g(x) = \log|T'(x)|$.

4. Examples

4.1 A family of unimodal maps. Consider the family of maps $T_l : [0,1] \to [0,1]$ defined by

$$T_l(x) = 1 - |2x-1|^l, \text{ for } l > 1.$$

For $l = 2$ this is the familiar Ulam-von Neumann map $T_2x = 4x(1-x)$, which is well-known to be conjugate to the strictly expanding tent map $S : I \to I$ defined by

$$S(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 1-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

This unimodal family has been well-studied, and it is known that each T_l has an absolutely continuous invariant measure μ_{T_l} (see for example [No-vS]). Recently Murray [M, p. 98] (see also [K-M-Y]) has shown that each map in this family is conjugate to an expanding Markov map, via the conjugating homeomorphism

$$\varphi_l(x) = \frac{1 + x^{1/l} - (1-x)^{1/l}}{2}.$$

This is a homeomorphism of I which fixes the three points 0, 1/2 and 1, is C^ω on $(0,1)$, and is non-differentiable at 0 and 1. We then define maps $S_l : I \to I$ by $S_l := \varphi_l \circ T_l \circ \varphi_l^{-1}$. Murray shows these maps are expanding, so there exist absolutely continuous invariant measures μ_{S_l}. The a.c.i.m.'s μ_{T_l} and μ_{S_l} are related by $\varphi_l^*\mu_{S_l} = \mu_{T_l}$. In particular, the corresponding invariant densities ρ_{T_l}, ρ_{S_l} are simply related by $\rho_{T_l}(x) = \rho_{S_l}(\varphi_l x)\varphi_l'(x)$.

The maps $S_l : [0,1] \to [0,1]$ are clearly analytic on the half-open intervals $(0, 1/2]$ and $[1/2, 1)$. (Of course, the individual analytic extension for each of these intervals need not, indeed will not, co-incide when they intersect).

The algebraic nature of the conjugacy means a little care is needed in proving analyticity of S_l at the points $x = 0, 1$. Let us write $\varphi_l(x) = x^{1/l}/2 + A(x)$, where A is analytic at 0, and $A(0) = 0$. The implicit function theorem implies that φ_l^{-1} is analytic at zero, and can be written $\varphi_l^{-1}(x) = 2^l x^l(1 + B(x))$, where B is analytic at zero with $B(0) = 0$. Near zero we can also write $T_l(x) = 2lx(1+C(x))$, where C is analytic at zero with $C(0) = 0$. Substituting into $S_l = \varphi_l \circ T_l \circ \varphi_l^{-1}$ we see that

$$S_l(x) \;=\; \varphi_l\left(2l\left[2^l x^l\left(1 + B\left(x\right)\right)\right]\left(1 + C\left(2^l x^l\left(1 + B(x)\right)\right)\right)\right).$$

Letting

$$D(x) = 2l\left[2^l x^l\left(1 + B\left(x\right)\right)\right]\left(1 + C\left(2^l x^l\left(1 + B(x)\right)\right)\right)$$

we have

$$S_l(x) = (2l)^{1/l}\, x\left(1 + B\left(x\right)\right)^{1/l}\left(1 + C\left(2^l x^l\left(1 + B(x)\right)\right)\right)^{1/l} + A\left(D(x)\right), \tag{4.1}$$

which is indeed analytic at $x = 0$. A similar argument applies at the point $x = 1$.

The conjugacy φ_l effects an isomorphism of the two measure-theoretic dynamical systems (T_l, μ_{T_l}) and (S_l, μ_{S_l}). In particular the metric entropies $h(\mu_{T_l})$ and $h(\mu_{S_l})$ are the same, so we will work with the expanding map S_l (to which our theory applies). In fact the conjugacy φ_l is differentiable except at the two endpoints of I. Therefore, with the exception of the fixed point at zero, the Lyapunov exponents of all periodic orbits of S_l agree with those of T_l (which are easier to compute). The Lyapunov exponent at the fixed point 0 of S_l is simply $\log(2l)^{1/l}$ (see (4.1)).

Murray [M, p. 101] estimates the metric entropy of the a.c.i.m. for T_4 using Ulam's method. He obtains $h(\mu_{T_4}) = 0 \cdot 684494 \pm 0 \cdot 004475$ using an approximation involving estimates on 20,480 subintervals.

Our calculations corroborate this, as shown in Table 1, which illustrates the rate at which our approximations converge.

M	$\int \log \lvert T_4' \rvert d\mu_M$
10	0.68449357504752271773574313505618766157945930
11	0.68449357504752260251509177842177235719158843
12	0.68449357504752260256820776598083056164855874
13	0.68449357504752260256819472702814929076462601
14	0.68449357504752260256819472868164339401607398
15	0.68449357504752260256819472868154757763010379
16	0.68449357504752260256819472868154757953058421

Table 1. Successive approximations to the entropy of the a.c.i.m. for $T_4(x) = 1 - |2x - 1|^4$

In Table 2 we list approximations to the entropy of the a.c.i.m. for T_l, $l = 3, 4, \ldots, 10$. All these approximations were derived using the first sixteen terms

l	$\int \log \lvert T_l' \rvert d\mu_{16}$
3	0.690856933490114499193735396741139191992 1
4	0.68449357504752260256819472868154757
5	0.675691061326954770979602536 43
6	0.66565877317301446916488 98
7	0.655103028496451495992
8	0.64442445688151295 04
9	0.63384949603792114
10	0.623504776625946

TABLE 2. Approximate entropy of the a.c.i.m. for $T_l(x) = 1 - |2x-1|^l$

of the (partial derivatives of the) Fredholm determinant. The quality of the approximation is better for lower values of l, reflecting the fact that the conjugate expanding map S_l has a larger analytic extension.

4.2 Perturbations of the doubling map. In section 2 we considered the family of maps $T_\varepsilon : [0,1] \to [0,1]$ defined by

$$T_\varepsilon(x) = 2x + \varepsilon \sin 2\pi x \pmod 1, \text{ for } -\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}.$$

Murray [M] considers a similar family of maps

$$S_\Omega(x) = 2x + \Omega + \frac{\sin 2\pi x}{4\pi} \pmod 1.$$

Using Ulam's method, with a partition of 10,000 subintervals, he estimates [M, p. 81] the entropies of the a.c.i.m.'s in this family to within an accuracy of $0 \cdot 0411$. The map $T_{1/4\pi} = S_0$ is common to both families, so let us compute the entropy of the a.c.i.m. for this map. We will also compare our method of approximation (using the measures μ_M) to that obtained by using the measures m_M defined in (0.1).

We remarked in the introduction that the convergence arising from the measures m_M is merely exponential. The approximate entropies $h_M = \int \log |T'| \, dm_M$ are given by

$$h_M = \frac{\sum_{x \in \mathrm{Fix}(M)} \log |T'(x)| / |(T^M)'(x)|}{\sum_{x \in \mathrm{Fix}(M)} 1/|(T^M)'(x)|}. \tag{4.2}$$

The reason for the comparatively slow convergence is that these expressions h_M do not fully exploit *all* periodic points up to period M, but rather are defined only in terms of points of period exactly M. Thus each length-M cylinder set (in the corresponding subshift of finite type) is represented by the unique period-M point lying within it. However any other consistent choice of representative would result in the same order of convergence (though the corresponding measures would no longer be invariant). Indeed by fixing some point y and summing over its pre-images under T^M, we can define approximations

$$H_M = \frac{\sum_{T^M x = y} \log |T'(x)| / |(T^M)'(x)|}{\sum_{T^M x = y} 1/|(T^M)'(x)|}, \tag{4.3}$$

which will converge to the true entropy at the same rate as the h_M. But

$$H_M = \frac{\mathcal{L}_T^M(\log|T'|)(y)}{\mathcal{L}_T^M(1)(y)} \longrightarrow \frac{\rho(y)\int \log|T'|\,d\mu}{\rho(y)\int 1\,d\mu} = h(\mu), \tag{4.4}$$

where the convergence in (4.4) is at exponential rate $O(\gamma^M)$, with γ the second largest eigenvalue of $\mathcal{L}_T$ (see [Ru2]).

In Table 3 we compare, for the map $T_{1/4\pi} = S_0$, the approximate entropies given by the two sequences of measures μ_M (super-exponential convergence) and m_M (exponential convergence).

M	$\int \log\lvert T'_{1/4\pi}\rvert\,d\mu_M$	$\int \log\lvert T'_{1/4\pi}\rvert\,dm_M$
6	0.68377196111065572152584683874828	0.684006
7	0.68377196024189881811155811511344	0.683903
8	0.68377196024214511529529028557809	0.683836
9	0.68377196024214513962679407279747	0.683805
10	0.68377196024214513961916059090822	0.683788
11	0.68377196024214513961916071092967	0.683780
12	0.68377196024214513961916071100328	0.683776

TABLE 3. Successive approximations to the entropy of the a.c.i.m. for $T_{1/4\pi}(x) = 2x + \frac{1}{4\pi}\sin(2\pi x)$

5. SOME FINAL COMMENTS

Our method also applies to real-analytic expanding Markov maps in higher dimensions. Interesting examples are the various continued fraction algorithms, such as the Jacobi-Perron algorithm, designed to give simultaneous diophantine approximations. In dimension d the approximation rate for the metric entropy and the Fourier coefficients is $O(\theta^{M^{1+1/d}})$.

If we relax the analyticity requirement, and demand only that T be C^2, say, then the same method applies, and all the rates of approximation will be exponential.

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