M. Sci. Examination by Course Unit 2011

MTH716U Measure Theory & Probability

Duration: 3 hours

Date and time: ??

Solutions

Question 1 (a) [3 marks]

Outer measure $m^*(A)$ is defined as

$$m^*(A) = \inf_{\{I_n\}} \sum_{n=1}^{\infty} l(I_n)\,,$$

the infimum being taken over all sequences $\{I_n\}$ of intervals satisfying $A \subseteq \bigcup_{n=1}^{\infty} I_n$, and where $l(I)$ denotes the length of an interval I.

(b) [6 marks]

If the right-hand side is infinite then the inequality clearly holds; so suppose $\sum_{n=1}^{\infty} m^*(A_n) < \infty$.

For any $\varepsilon > 0$ and $n \geq 1$ there is a sequence of intervals $\{I_k^n\}_{k=1}^{\infty}$ covering A_n such that

$$\sum_{k=1}^{\infty} l(I_k^n) \leq m^*(A_n) + \frac{\varepsilon}{2^n}\,.$$

So the countable collection of intervals $\{I_k^n\}_{k,n\geq 1}$ covers $\bigcup_{n=1}^{\infty} A_n$, thus

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} l(I_k^n)$$

and also

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} l(I_k^n) \leq \sum_{n=1}^{\infty}\left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon\,.$$

So

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon\,,$$

and letting $\varepsilon \to 0$ completes the proof.

(c) [3 marks]

Let C_0 denote the closed unit interval. For $n \geq 1$, define C_n to be the union of 2^n disjoint closed intervals obtained by removing the open central interval of length $1/3^n$ from each of the 2^{n-1} disjoint closed intervals whose union is C_{n-1}. Then define $C = \cap_{n\geq 0} C_n$.

(d) [4 marks]

For any $n \geq 0$, define the intervals $\{I_i^n\}_{i=1}^{\infty}$ by: $I_1^n, \ldots, I_{2^n}^n$ is an enumeration of the 2^n disjoint closed intervals whose union is C_n, and $I_i^n = \emptyset$ for $i > 2^n$. Clearly $\{I_i^n\}_{i=1}^{\infty}$ is a cover of C (since it is a cover of C_n), and $\sum_{i\geq 1} l(I_i^n) = 2^n(1/3)^n \to 0$ as $n \to \infty$. So C can be covered by a sequence of intervals of arbitarily small total length, i.e. $m^*(C) = 0$.

(e) [3 marks]

A subset $E \subset \mathbb{R}$ is measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for all subsets $A \subseteq \mathbb{R}$.

[Full marks too for giving the condition as simply $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$, since the reverse inequality always holds.]

(f) [3 marks]

A collection of subsets of a non-empty set is a σ-field if it contains the empty set, and is closed under complementation and countable union.

(g) [3 marks]

Clearly $(-\infty, \infty) = \mathbb{R} = \emptyset^c \in \mathscr{M}$, so it remains to check that for every $a \in \mathbb{R}$, the intervals $(-\infty, a)$, $(-\infty, a]$, (a, ∞) and $[a, \infty)$ lie in $\mathscr{M}$.

Now $(-\infty, a) = \cup_{n \geq 1}(a-n, a)$ and $(-\infty, a] = \cup_{n \geq 1}(a-n, a]$ and $(a, \infty) = \cup_{n \geq 1}(a, a+n)$ and $[a, \infty) = \cup_{n \geq 1}[a, a+n)$ are countable unions of members of $\mathscr{M}$ (namely, finite-length intervals), and $\mathscr{M}$ is closed under countable union, so indeed each of these intervals belongs to $\mathscr{M}$.

Question 2 (a) [3 marks]

$f : \mathbb{R} \to \mathbb{R}$ is *measurable* if for every interval I, the set $f^{-1}(I)$ is measurable.

(b) [3 marks]

One possible equivalent condition is that $f^{-1}([a,\infty)) \in \mathscr{M}$ for every $a \in \mathbb{R}$.

(c) [5 marks]

If f is increasing then for each $a \in \mathbb{R}$, the set $f^{-1}([a,\infty))$ is either empty (if $a > f(x)$ for all $x \in \mathbb{R}$), or the interval $[b,\infty)$ or (b,∞) for some $b(= \inf\{x \in \mathbb{R} : f(x) \geq a\})$. These are all measurable sets, so f is measurable.

(d) [7 marks]

First suppose that n is even. If $a < 0$ then $(f^n)^{-1}([a,\infty)) = \mathbb{R}$, a measurable set. If $a \geq 0$ then $(f^n)^{-1}([a,\infty)) = \{x : f(x) \geq a^{1/n}\} \cup \{x : f(x) \leq -a^{1/n}\} = f^{-1}[a^{1/n},\infty) \cup f^{-1}(-\infty, -a^{1/n}]$, which is measurable because it is the union of two sets, both of which are measurable due to the measurability of f. Therefore f^n is measurable.

Now suppose n is odd. If $a \geq 0$ then $(f^n)^{-1}([a,\infty)) = \{x : f(x) \geq a^{1/n}\} = f^{-1}[a^{1/n},\infty)$, which is measurable because f is. If $a < 0$ then $(f^n)^{-1}([a,\infty)) = \{x : f(x) \geq -|a|^{1/n}\} = f^{-1}[-|a|^{1/n},\infty)$, which is measurable because f is. Therefore f^n is measurable.

(e) [4 marks]

The (standard) example given in class is the following. Define an equivalence relation on $[0,1]$ by $x \sim y$ if $y - x$ is rational. This partitions $[0,1]$ into disjoint equivalence classes. Now use the axiom of choice to construct a new set $E \subset [0,1]$ which contains exactly one member from each equivalence class. The set E can be shown to be non-measurable.

(f) [3 marks]

Let A be a non-measurable set (for example the one of part (e)), and define f by $f(x) = 1$ for $x \in A$, and $f(x) = -1$ for $x \notin A$. Then f is non-measurable, since e.g. $f^{-1}[1,\infty) = A$ is not measurable. But $f^2 \equiv 1$, a constant function, hence f^2 is measurable.

Question 3 (a) [3 marks]

A (non-negative) function $\varphi : \mathbb{R} \to \mathbb{R}$ is *simple* if it is measurable and only takes finitely many values, i.e. if $\varphi(\mathbb{R}) = \{a_1, \dots, a_n\}$ for some $a_1, \dots, a_n \in \mathbb{R}$.

(b) [3 marks]

If $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ then

$$\int_E \varphi \, dm = \sum_{i=1}^n a_i m(A_i \cap E)\,.$$

(c) [3 marks]

$$\int_E f \, dm = \sup_{\varphi} \int_E \varphi \, dm\,,$$

where the supremum is taken over those simple functions φ satisfying $0 \leq \varphi(x) \leq f(x)$ for all $x \in E$.

(d) [4 marks]

Fatou's Lemma states that if $\{f_n\}$ is a sequence of non-negative measurable functions (defined on a measurable set E), then

$$\liminf_{n\to\infty} \int_E f_n \, dm \ \geq \ \int_E (\liminf_{n\to\infty} f_n) \, dm\,.$$

(e) [4 marks]

The Monotone Convergence Theorem asserts that if $\{f_n\}$ is a sequence of non-negative measurable functions (defined on a measurable set E), and the sequence $\{f_n(x)\}_{n=1}^\infty$ increases monotonically to $f(x)$ for each $x \in E$, then

$$\lim_{n\to\infty} \int_E f_n \, dm \ = \ \int_E f \, dm\,.$$

(f) [8 marks]

Now $f_n \leq f$, so $\int_E f_n \, dm \leq \int_E f \, dm$, hence

$$\limsup_{n\to\infty} \int_E f_n \, dm \leq \int_E f \, dm\,.$$

Fatou's Lemma gives

$$\int_E f \, dm \ \leq \ \liminf_{n\to\infty} \int_E f_n \, dm\,,$$

and clearly also

$$\liminf_{n\to\infty} \int_E f_n \, dm \ \leq \ \limsup_{n\to\infty} \int_E f_n \, dm\,.$$

Combining the above 3 inequalities gives

$$\int_E f \, dm = \liminf_{n\to\infty} \int_E f_n \, dm = \limsup_{n\to\infty} \int_E f_n \, dm\,,$$

so the sequence $\int_E f_n \, dm$ converges to $\int_E f \, dm$, as required.

Question 4 (a) [4 marks]

The Dominated Convergence Theorem asserts the following: Suppose $E \subset \mathbb{R}$ is measurable. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions (defined on E) such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and almost every $x \in E$, for some function g which is integrable over E. If $f = \lim_{n\to\infty} f_n$ almost everywhere then f is integrable over E, and

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm .$$

(b) [5 marks]

For each $n \geq 1$ and $x \in [1, \infty)$ we have

$$|f_n(x)| = f_n(x) == \frac{x \sin \pi n x}{1 + n x^3} \leq \frac{x}{n x^3} \leq x^{-2} =: g(x) .$$

Since g is integrable on $[1, \infty)$, and $f_n \to 0$ pointwise, the Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \int_1^{\infty} f_n \, dm = \int_1^{\infty} 0 \, dm = 0 .$$

(c) [4 marks]

Beppo Levi's Theorem asserts the following:

Suppose $f_1, f_2, \ldots$ are measurable, and $\sum_{k=1}^{\infty} \int |f_k| \, dm < \infty$.

Then the series $\sum_{k=1}^{\infty} f_k(x)$ converges for almost every x, it defines an integrable function, and $\int \sum_{k=1}^{\infty} f_k \, dm = \sum_{k=1}^{\infty} \int f_k \, dm$.

(d) [7 marks]

The function $\varphi(x) := \sum_{k=1}^{\infty} |f_k(x)|$ is non-negative and measurable, and by applying the Monotone Convergence Theorem to the sequence of partial sums $\sum_{k=1}^{n} |f_k|$ we see that

$$\int \varphi \, dm = \sum_{k=1}^{\infty} \int |f_k| \, dm .$$

The righthand side of this is finite by hypothesis, so φ is integrable.

Consequently φ is finite almost everywhere, i.e. the series $\sum_{k=1}^{\infty} |f_k(x)|$ converges for almost every x.

Hence the series $\sum_{k=1}^{\infty} f_k(x)$ converges for almost every x (as required).

Define $f : \mathbb{R} \to \mathbb{R}$ (almost everywhere) by $f(x) = \sum_{k=1}^{\infty} f_k(x)$.

For every $n \geq 1$ we have $\sum_{k=1}^{n} f_k \leq \varphi$, so the Dominated Convergence Theorem tells us that $f = \lim_{n\to\infty} \sum_{k=1}^{n} f_k$ is integrable (as required), and that

$$\int f \, dm = \int \lim_{n\to\infty} \sum_{k=1}^{n} f_k \, dm = \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \, dm = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, dm = \sum_{k=1}^{\infty} \int f_k \, dm ,$$

as required.

(e) [5 marks]

First note that

$$\frac{x}{e^x - 1} = x\frac{e^{-x}}{1 - e^{-x}} = x\sum_{n=1}^{\infty} e^{-nx} = \sum_{n=1}^{\infty} xe^{-nx}.$$

Integration by parts gives

$$\int_0^\infty xe^{-nx}\,dx = x(-1/n)e^{-nx}\Big|_0^\infty \; - \; (-1/n)\int_0^\infty e^{-nx}\,dx = 1/n^2.$$

Defining $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = xe^{-nx}$ for $x \geq 0$ and $f_n(x) = 0$ for $x < 0$ we see that the hypotheses of Beppo Levi's Theorem are satisfied, and $\int f_n\,dm = 1/n^2$, so

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \int \sum_{n=1}^{\infty} f_n\,dm = \sum_{n=1}^{\infty} \int f_n\,dm = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

as required.

 END OF EXAMINATION