Queen Mary
University of London

M. Sci. Examination by course unit 2011

MTH716U   Measure Theory & Probability

Duration: 3 hours

Date and time: 20th May 2011,   1000h–1300h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): O.M.Jenkinson

Standing Assumption: Throughout this exam, the term *measurable* means *Lebesgue measurable*.

Question 1 (a) [3 marks] For a subset $A \subseteq \mathbb{R}$, how is its *outer measure* $m^*(A)$ defined?

(b) [6 marks] Use the definition of m^* to prove that if $A_1, A_2, A_3, \ldots$ are subsets of $\mathbb{R}$, then

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

(c) [3 marks] How is the middle-third Cantor set C defined?

(d) [4 marks] Prove that the middle-third Cantor set C satisfies $m^*(C) = 0$.

(e) [3 marks] What does it mean to say that a subset of $\mathbb{R}$ is *measurable*?

(f) [3 marks] How is a σ-*field* defined?

(g) [3 marks] Let $\mathscr{M}$ denote the σ-field consisting of measurable subsets of $\mathbb{R}$. Assuming that all finite-length intervals are measurable, prove that every interval is measurable.

Question 2 (a) [3 marks] What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is *measurable*?

(b) [3 marks] State, but do not prove, a condition equivalent to the statement that $f : \mathbb{R} \to \mathbb{R}$ is measurable. (You may wish to choose this condition to be one which can be used in (c) and (d) below.)

(c) [5 marks] Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing (i.e. $f(x) \leq f(y)$ whenever $x \leq y$) then it is measurable.

(d) [7 marks] Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Use the definition of measurability to prove that for every integer $n \geq 2$, the function f^n defined by $(f^n)(x) = (f(x))^n$ is also measurable. [Note: In your proof you should *not* use the fact that the product of measurable functions is measurable.]

(e) [4 marks] Assuming the Axiom of Choice, briefly describe (without proof) the construction of a non-measurable set.

(f) [3 marks] Using your answer to part (e), or otherwise, give an example of a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^2 : \mathbb{R} \to \mathbb{R}$ is measurable (where f^2 is defined by $(f^2)(x) = f(x)^2$).

Question 3 (a) [3 marks] What does it mean to say that a non-negative function $\varphi : \mathbb{R} \to \mathbb{R}$ is *simple*?

(b) [3 marks] For a measurable subset $E \subseteq \mathbb{R}$, and simple function $\varphi : \mathbb{R} \to \mathbb{R}$, how is the *(Lebesgue) integral* $\int_E \varphi \, dm$ defined?

(c) [3 marks] If $f : \mathbb{R} \to \mathbb{R}$ is non-negative and measurable, and $E \subseteq \mathbb{R}$ is measurable, how is the *(Lebesgue) integral* $\int_E f \, dm$ defined?

(d) [4 marks] State Fatou's Lemma for a sequence of measurable functions.

(e) [4 marks] State the Monotone Convergence Theorem.

(f) [8 marks] Prove that Fatou's Lemma implies the Monotone Convergence Theorem.

Question 4 (a) [4 marks] State the Dominated Convergence Theorem.

(b) [5 marks] Use the Dominated Convergence Theorem to find

$$\lim_{n\to\infty} \int_1^\infty f_n \, dm ,$$

where for each $n \geq 1$ the function $f_n : [1, \infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{x \sin \pi n x}{1 + n x^3} .$$

(c) [4 marks] State Beppo Levi's Theorem.

(d) [7 marks] Use the Dominated Convergence Theorem to prove Beppo Levi's Theorem.

(e) [5 marks] Use Beppo Levi's Theorem, and the fact that $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, to prove that

$$\int_0^\infty \frac{x}{e^x - 1} \, dx = \frac{\pi^2}{6} .$$

End of Paper