Queen Mary
University of London

M. Sci. Examination by course unit 2010

MTH716U Measure Theory & Probability

Duration: 3 hours

Date and time: 19th May 2010, 1000h–1300h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): O.M.Jenkinson

Standing Assumption: Throughout this exam, the term *measurable* means *Lebesgue measurable*.

Question 1 (a) [3 marks] For a subset $A \subseteq \mathbb{R}$, how is its *outer measure* $m^*(A)$ defined?

(b) [6 marks] Use the definition of m^* to show that if A_1 and A_2 are subsets of $\mathbb{R}$, then

$$m^*(A_1 \cup A_2) \leq m^*(A_1) + m^*(A_2).$$

(c) [6 marks] Show that for any subset $A \subset \mathbb{R}$, and any $t \in \mathbb{R}$,

$$m^*(A+t) = m^*(A),$$

where $A+t := \{a+t : a \in A\}$ denotes the translation of A by t.

(d) [2 marks] Given a set Ω, what is meant by a σ*-field* $\mathscr{F}$ (of subsets of Ω)?

(e) [2 marks] What is meant by a *measure space* $(\Omega, \mathscr{F}, \mu)$?

(f) [6 marks] Let $\Omega = \mathbb{Z}$ be the set of integers, and 2^Ω its power set (i.e. the collection of all subsets of Ω). For $A \subset \Omega$, let $\mu(A)$ denote the number of elements in A if A is finite, and $\mu(A) = \infty$ otherwise. Show that $(\Omega, 2^\Omega, \mu)$ is a measure space.

Question 2 (a) [3 marks] What does it mean to say that a subset of $\mathbb{R}$ is *measurable*?

(b) [2 marks] What does it mean to say that a subset of $\mathbb{R}$ is a *null set*?

(c) [5 marks] Prove that every null set is measurable.

(d) [4 marks] Briefly describe (without proof) the construction of an uncountable subset of $\mathbb{R}$ which is null.

(e) [4 marks] Assuming the Axiom of Choice, briefly describe (without proof) the construction of a non-measurable set.

(f) [7 marks] Let m be Lebesgue measure. Assuming that $m\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} m(A_n)$ whenever $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ are measurable, prove that if $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ are measurable and $m(B_1) < \infty$, then

$$m\left(\bigcap_{n=1}^\infty B_n\right) = \lim_{n\to\infty} m(B_n).$$

Question 3 (a) [3 marks] What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is *measurable*?

(b) [3 marks] State, but do not prove, a condition equivalent to the statement that $f : \mathbb{R} \to \mathbb{R}$ is measurable. (You may wish to choose this condition to be one which can be used in (c), (d) and (f) below).

(c) [7 marks] Using your answer to (b), or otherwise, show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both measurable, then so is the function $f + g$.

(d) [5 marks] Using your answer to (b), or otherwise, show that if $f_n : \mathbb{R} \to \mathbb{R}$ is measurable for each $n \geq 1$, then so is $\sup_{n \geq 1} f_n$.

(e) [2 marks] What does it mean to say that two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are *equal almost everywhere*?

(f) [5 marks] Using your answer to (b), or otherwise, show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable, and is almost everywhere equal to another function $g : \mathbb{R} \to \mathbb{R}$, then g is also measurable.

Question 4 (a) [4 marks] State Fatou's Lemma for a sequence of measurable functions.

(b) [3 marks] Give an example of a sequence of measurable functions such that the conclusion of Fatou's Lemma is a *strict* inequality.

(c) [4 marks] State the Dominated Convergence Theorem.

(d) [10 marks] Prove that Fatou's Lemma implies the Dominated Convergence Theorem.

(e) [4 marks] Use the Dominated Convergence Theorem to show that if $f : \mathbb{R} \to \mathbb{R}$ is integrable, and the function $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) := \min(f(x), n)$ for each $n \geq 1$, $x \in \mathbb{R}$, then $\int f_n \, dm \to \int f \, dm$ as $n \to \infty$.

End of Paper