M. Sci. Examination by Course Unit 2010

MTH716U Measure Theory & Probability

Duration: 3 hours

Date and time: 19th May 2010

Solutions

Question 1 (a) Outer measure $m^*(A)$ is defined as

$$m^*(A) = \inf_{\{I_n\}} \sum_{n=1}^{\infty} l(I_n)\,,$$

the infimum being taken over all sequences $\{I_n\}$ of intervals satisfying $A \subseteq \bigcup_{n=1}^{\infty} I_n$, and where $l(I)$ denotes the length of an interval I.

(b) It suffices to show that for any $\varepsilon > 0$,

$$m^*(A_1 \cup A_2) \leq m^*(A_1) + m^*(A_2) + \varepsilon\,.$$

Let $\{I_n^1\}$ be a sequence of intervals covering A_1, and $\{I_n^2\}$ a sequence of intervals covering A_2, such that

$$\sum_{n=1}^{\infty} l(I_n^1) \leq m^*(A_1) + \varepsilon/2$$

and

$$\sum_{n=1}^{\infty} l(I_n^2) \leq m^*(A_2) + \varepsilon/2\,.$$

But the sequence of intervals $\{I_1^1, I_1^2, I_2^1, I_2^2, I_3^1, \ldots\}$ covers $A_1 \cup A_2$, so

$$m^*(A_1 \cup A_2) \leq \sum_{n=1}^{\infty} l(I_n^1) + \sum_{n=1}^{\infty} l(I_n^2) \leq m^*(A_1) + m^*(A_2) + \varepsilon\,,$$

as required.

(c) If a sequence of intervals $\{I_n\}$ covers A then the sequence of translated intervals $\{I_n + t\}$ covers $A + t$. Conversely, if the sequence $\{J_n\}$ covers $A + t$ then the sequence $\{J_n - t\}$ covers A. Moreover, the total length of a sequence of intervals does not change when we translate each interval by a common number.

So we have a one-to-one correspondence between the interval coverings of A and $A + t$, and this correspondence preserves the total length of the covering. Thus the infimum of these total lengths, taken over the set of interval coverings, is the same in both cases. It follows from the definition of m^* that $m^*(A) = m^*(A + t)$, as required.

(d) It means that $\mathscr{F}$ contains the empty set, and is closed under complementation and countable unions.

(e) It means that $\mathscr{F}$ is a σ-field, and that μ is a measure, i.e. that $\mu : \mathscr{F} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$, and if $A_1, A_2 \ldots$ are pairwise disjoint members of $\mathscr{F}$ then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

(f) Clearly 2^{Ω} is a σ-field, so we need only show that μ is a measure.

Certainly $\mu(\emptyset) = 0$, so it remains to check that if $A_1, A_2 \dots$ are pairwise disjoint then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If there exists N such that $A_i = \emptyset$ for all $i > N$, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i) .$$

If such an N does not exist then, because the A_i are pairwise disjoint, $\cup_{i=1}^{\infty} A_i$ is an infinite set, so $\mu(\cup_{i=1}^{\infty} A_i) = \infty$. But also $\mu(A_i) \geq 1$ for infinitely many distinct i, so $\sum_{i=1}^{\infty} \mu(A_i) = \infty = \mu(\cup_{i=1}^{\infty} A_i)$.

Question 2 (a) A subset $E \subset \mathbb{R}$ is measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for all subsets $A \subseteq \mathbb{R}$.

[Full marks too for giving the condition as simply $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$, since the reverse inequality always holds.]

(b) $N \subseteq \mathbb{R}$ is a null set if $m^*(N) = 0$, i.e. if for every $\varepsilon > 0$ we can find a sequence $\{I_n\}$ of intervals satisfying $N \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) < \varepsilon$.

(c) If N is a null set then $m^*(N) = 0$, so for any $A \subset \mathbb{R}$ we have

$$m^*(A \cap N) \leq m^*(N) = 0,$$

and

$$m^*(A \cap N^c) \leq m^*(A),$$

so adding these inequalities gives

$$m^*(A \cap N) + m^*(A \cap N^c) \leq m^*(A),$$

thus N is measurable.

(d) One example is the middle-third Cantor set. This is the set $A = \bigcap_{n=1}^{\infty} A_n$, where $A_0 = [0,1]$, and in general A_{n+1} denotes the disjoint union of 2^{n+1} closed intervals obtained by removing the (open) 'middle third' from each of the 2^n disjoint closed intervals whose union is A_n.

(e) The (standard) example given in class is the following. Define an equivalence relation on $[0,1]$ by $x \sim y$ if $y - x$ is rational. This partitions $[0,1]$ into disjoint equivalence classes. Now use the axiom of choice to construct a new set $E \subset [0,1]$ which contains exactly one member from each equivalence class. The set E can be shown to be non-measurable.

(f) The sequence $\{B_1 \setminus B_n\}_{n=1}^{\infty}$ is increasing, so the assumption gives

$$m\left(\bigcup_{n=1}^{\infty} B_1 \setminus B_n\right) = \lim_{n\to\infty} m(B_1 \setminus B_n).$$

But $m(B_1) < \infty$, so $m(B_1 \setminus B_n) = m(B_1) - m(B_n)$, thus

$$m\left(\bigcup_{n=1}^{\infty} B_1 \setminus B_n\right) = m(B_1) - \lim_{n\to\infty} m(B_n).$$

Now $\bigcup_{n=1}^{\infty} B_1 \setminus B_n = B_1 \setminus \bigcap_{n=1}^{\infty} B_n$, so

$$m\left(\bigcup_{n=1}^{\infty} B_1 \setminus B_n\right) = m(B_1 \setminus \bigcap_{n=1}^{\infty} B_n) = m(B_1) - m(\bigcap_{n=1}^{\infty} B_n)$$

(using that $m(B_1) < \infty$ again).

Combining the previous two displayed equations gives

$$m(B_1) - \lim_{n\to\infty} m(B_n) = m(B_1) - m(\bigcap_{n=1}^{\infty} B_n),$$

hence $m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n)$ as required.

Question 3 (a) $f : \mathbb{R} \to \mathbb{R}$ is *measurable* if for every interval I, the set $f^{-1}(I)$ is measurable.

(b) One possible equivalent condition is: $f^{-1}((a,\infty))$ is measurable for every $a \in \mathbb{R}$.

(c) We must show that

$$B := (f+g)^{-1}((a,\infty))$$

is a measurable set, for every $a \in \mathbb{R}$.

Let $\{q_n\}$ be an enumeration of the rationals. We claim that

$$B = \bigcup_{n=1}^{\infty} \{t : f(t) > q_n,\ g(t) > a - q_n\}\,. \tag{1}$$

To see that (1) is indeed true, note that if t belongs to the set on the righthand side then for some n we have $f(t) > q_n$ and $g(t) > a - q_n$, so $f(t) + g(t) > a$, so $t \in B$.

Conversely, if $t \in B$ then $(f+g)(t) > a$, so $f(t) > a - g(t)$, so we can choose a rational q_n such that $f(t) > q_n > a - g(t)$, so t belongs to the set on the righthand side of (1). So (1) does indeed hold.

Now the righthand side of (1) is a countable union of sets

$$\{t : f(t) > q_n,\ g(t) > a - q_n\} = f^{-1}((q_n,\infty)) \cap g^{-1}((a-q_n,\infty))$$

which are measurable because f and g are. Therefore B is a measurable set, as required.

(d) Note that for any $a \in \mathbb{R}$,

$$(\sup_{n\geq 1} f_n)^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty))$$

is a measurable set, since it is a union of sets $f_n^{-1}((a,\infty))$ which are measurable because f_n is. Therefore $\sup_{n\geq 1} f_n$ is a measurable function.

(e) It means that there is some null set A such that $f(x) = g(x)$ for all $x \in \mathbb{R} \setminus A$.

(f) Consider the difference function $d = g - f$. It is zero except on a null set, so $d^{-1}((a,\infty))$ is a null set if $a \geq 0$, and is the complement of a null set if $a < 0$. Both null sets and their complements are measurable, hence d is measurable. Therefore $g = d + f$ is measurable, as it is the sum of two measurable functions.

Question 4 (a) Fatou's Lemma states that if $\{f_n\}$ is a sequence of non-negative measurable functions (defined on a measurable set E), then

$$\liminf_{n\to\infty} \int_E f_n\, dm \ \geq\ \int_E (\liminf_{n\to\infty} f_n)\, dm\,.$$

(b) One example is to choose $f_n : (0,1) \to \mathbb{R}$ to equal the constant value n on the sub-interval $(0,1/n)$, and the constant value 0 otherwise. In this case each $\int f_n\, dm = 1$, yet $f_n \to 0$ pointwise, so $\int (\liminf_{n\to\infty} f_n)\, dm = 0$.

(c) The Dominated Convergence Theorem asserts the following: Suppose $E \subset \mathbb{R}$ is measurable. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions (defined on E) such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and almost every $x \in E$, for some function g which is integrable over E. If $f = \lim_{n\to\infty} f_n$ almost everywhere then f is integrable over E, and

$$\lim_{n\to\infty} \int_E f_n\, dm \ =\ \int_E f\, dm\,.$$

(d) Consider first the case where each $f_n \geq 0$. Fatou's Lemma gives:

$$\int_E f\, dm \leq \liminf_{n\to\infty} \int_E f_n\, dm\,.$$

It is therefore sufficient to show that

$$\limsup_{n\to\infty} \int_E f_n\, dm \leq \int_E f\, dm\,. \tag{2}$$

Fatou's Lemma applied to $g - f_n$ gives

$$\int_E \lim_{n\to\infty} (g - f_n)\, dm \leq \liminf_{n\to\infty} \int_E (g - f_n)\, dm\,.$$

On the left we have

$$\int_E \lim_{n\to\infty} (g - f_n)\, dm = \int_E (g - f)\, dm = \int_E g\, dm - \int_E f\, dm\,,$$

while on the right we have

$$\liminf_{n\to\infty} \int_E (g - f_n)\, dm = \liminf_{n\to\infty} \left(\int_E g\, dm - \int_E f_n\, dm \right) = \int_E g\, dm - \limsup_{n\to\infty} \int_E f_n\, dm\,.$$

Combining these we get:

$$\int_E g\, dm - \int_E f\, dm \leq \int_E g\, dm - \limsup_{n\to\infty} \int_E f_n\, dm\,.$$

Finally, subtract $\int_E g\, dm$ (which is finite) and multiply by -1 to obtain (2).

Now consider a general (not necessarily negative) sequence $\{f_n\}$. Since by hypothesis

$$-g(x) \leq f_n(x) \leq g(x)$$

we have

$$0 \leq f_n(x) + g(x) \leq 2g(x)\,,$$

so applying the result for non-negative functions to the sequence $f_n + g$ (since the function $2g$ is certainly integrable) gives the result.

(e) Clearly $f_n \to f$ pointwise. But the convergence is dominated: $|f_n(x)| = |f(x)|$ if $f(x) \leq n$, and $|f_n(x)| = n \leq f(x) = |f(x)|$ if $f(x) \geq n$, so $|f_n| \leq |f|$, and $|f|$ is integrable (because f is). The Dominated Convergence Theorem then implies that $\int f_n \, dm \to \int f \, dm$ as $n \to \infty$.

 END OF EXAMINATION