Queen Mary
University of London

M. Sci. Examination by course unit 2009

MTH716U Measure Theory & Probability

Duration: 3 hours

Date and time: 28 April 2009, 1000h–1300h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): O.M.Jenkinson

Question 1 (a) [4 marks] For a subset $A \subseteq \mathbb{R}$, how is its *(Lebesgue) outer measure* $m^*(A)$ defined?

(b) [2 marks] What does it mean to say that $A \subseteq \mathbb{R}$ is a *null set*?

(c) [3 marks] Briefly explain why every countable subset $A \subseteq \mathbb{R}$ is null.

(d) [4 marks] Briefly describe (without proof) the construction of an uncountable subset $A \subseteq \mathbb{R}$ which is null.

(e) [4 marks] Prove that if $A \subseteq B \subseteq \mathbb{R}$ then $m^*(A) \leq m^*(B)$.

(f) [8 marks] If $A_1, A_2, A_3, \ldots$ are subsets of $\mathbb{R}$, prove that

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)\,.$$

Question 2 (a) [3 marks] What does it mean to say that a subset $E \subseteq \mathbb{R}$ is *(Lebesgue-) measurable*?

(b) [3 marks] Show that if $E \subseteq \mathbb{R}$ is measurable, then so is its complement E^c $(= \mathbb{R} \setminus E)$.

(c) [2 marks] What does it mean to say that Lebesgue measure m is *countably additive*?

(d) [6 marks] Assuming countable additivity of Lebesgue measure m, show that if the sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ are measurable, then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)\,.$$

(e) [8 marks] Using (d) above, or otherwise, show that if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are measurable, and $m(A_1) < \infty$, then

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)\,.$$

(f) [3 marks] Give an example of measurable sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ such that

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) \neq \lim_{n\to\infty} m(A_n)\,.$$

Question 3 (a) [3 marks] What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is *(Lebesgue) measurable*?

(b) [3 marks] What does it mean to say that a non-negative function $\varphi : \mathbb{R} \to \mathbb{R}$ is *simple*?

(c) [3 marks] For a measurable subset $E \subseteq \mathbb{R}$, and simple function $\varphi : \mathbb{R} \to \mathbb{R}$, how is the *(Lebesgue) integral* $\int_E \varphi \, dm$ defined?

(d) [3 marks] If $f : \mathbb{R} \to \mathbb{R}$ is non-negative and measurable, and $E \subseteq \mathbb{R}$ is measurable, how is the *(Lebesgue) integral* $\int_E f \, dm$ defined?

(e) [4 marks] Let $E \subseteq \mathbb{R}$ be measurable, and suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are non-negative measurable functions, with $f(x) \leq g(x)$ for all $x \in E$. Show that $\int_E f \, dm \leq \int_E g \, dm$.

(f) [9 marks] If $f : \mathbb{R} \to \mathbb{R}$ is non-negative and measurable, prove that $\int_{\mathbb{R}} f \, dm = 0$ if and only if $f = 0$ almost everywhere.

Question 4 (a) [4 marks] State Fatou's Lemma for a sequence of (Lebesgue) measurable functions.

(b) [4 marks] State the Monotone Convergence Theorem.

(c) [8 marks] Prove that Fatou's Lemma implies the Monotone Convergence Theorem.

(d) [4 marks] State the Dominated Convergence Theorem.

(e) [5 marks] Use the Dominated Convergence Theorem to find

$$\lim_{n\to\infty} \int_1^\infty f_n \, dm ,$$

where for each $n \geq 1$ the function $f_n : [1,\infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{\sqrt{x}}{1+nx^4} .$$

End of Paper