Queen Mary
University of London

M. Sci. Examination by Course Unit 2009

MTH716U Measure Theory & Probability

Duration: 2 hours

Date and time: 28 April 2009

Solutions

Question 1 (a) (Lebesgue) outer measure $m^*(A)$ is defined as

$$m^*(A) = \inf_{\{I_n\}} \sum_{n=1}^{\infty} l(I_n) \,,$$

the infimum being taken over all sequences $\{I_n\}$ of intervals satisfying $A \subseteq \bigcup_{n=1}^{\infty} I_n$, and where $l(I)$ denotes the length of an interval I.

(b) $A \subseteq \mathbb{R}$ is a null set if $m^*(A) = 0$, i.e. if for every $\varepsilon > 0$ we can find a sequence $\{I_n\}$ of intervals satisfying $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) < \varepsilon$.

(c) If A is countable, with distinct elements $x_1, x_2, \ldots$ (the list can be finite or infinite), then we can write it as $A = \bigcup_{n=1}^{\infty} I_n$, where either I_n is the closed interval $[x_n, x_n]$ or the empty set ($I_n = (5,5)$, say). Now $l(I_n) = 0$ for each n, so $\sum_{n=1}^{\infty} l(I_n) = 0$, and clearly $A \subseteq \bigcup_{n=1}^{\infty} I_n$, so A is null.

(d) One example is the middle-third Cantor set. This is the set $A = \bigcap_{n=1}^{\infty} A_n$, where $A_0 = [0,1]$, and in general A_{n+1} denotes the disjoint union of 2^{n+1} closed intervals obtained by removing the (open) 'middle third' from each of the 2^n disjoint closed intervals whose union is A_n.

(e) If $A \subseteq B$ then any sequence of intervals $\{I_n\}$ satisfying $B \subseteq \bigcup_{n=1}^{\infty} I_n$ also satisfies $A \subseteq \bigcup_{n=1}^{\infty} I_n$. So the infimum in the definition of $m^*(A)$ is over a larger collection of sequences of intervals than the infimum in the definition of $m^*(B)$, hence $m^*(A) \leq m^*(B)$.

(f) If the right-hand side is infinite then the inequality clearly holds; so suppose $\sum_{n=1}^{\infty} m^*(A_n) < \infty$.

For any $\varepsilon > 0$ and $n \geq 1$ there is a sequence of intervals $\{I_k^n\}_{k=1}^{\infty}$ covering A_n such that

$$\sum_{k=1}^{\infty} l(I_k^n) \leq m^*(A_n) + \frac{\varepsilon}{2^n} \,.$$

So the countable collection of intervals $\{I_k^n\}_{k,n\geq 1}$ covers $\bigcup_{n=1}^{\infty} A_n$, thus

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(I_k^n)$$

and also

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(I_k^n) \leq \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon \,.$$

So

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon \,,$$

and letting $\varepsilon \to 0$ completes the proof.

Question 2 (a) E is (Lebesgue-) measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for all subsets $A \subseteq \mathbb{R}$.

(b) Now $(E^c)^c = E$, so if $A \subseteq \mathbb{R}$ and $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$, then

$$m^*(A \cap E^c) + m^*(A \cap (E^c)^c) = m^*(A \cap E^c) + m^*(A \cap E) = m^*(A)\,.$$

(c) Countable additivity means that if the measurable sets $B_1, B_2, \ldots$ are pairwise disjoint then $m(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} m(B_n)$.

(d) Let $B_1 = A_1$, and $B_i = A_i \setminus A_{i-1}$ for $i > 1$. Then each B_i is measurable, and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$, so countable additivity gives

$$m(\bigcup_{i=1}^{\infty} A_i) = m(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} m(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} m(B_i) = \lim_{n\to\infty} m(\bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} m(A_n)\,,$$

since $A_n = \bigcup_{i=1}^{n} B_i$.

(e) The sequence $\{A_1 \setminus A_n\}_{n=1}^{\infty}$ is increasing, so (d) gives

$$m\left(\bigcup_{n=1}^{\infty} A_1 \setminus A_n\right) = \lim_{n\to\infty} m(A_1 \setminus A_n)\,.$$

But $m(A_1) < \infty$, so $m(A_1 \setminus A_n) = m(A_1) - m(A_n)$, thus

$$m\left(\bigcup_{n=1}^{\infty} A_1 \setminus A_n\right) = m(A_1) - \lim_{n\to\infty} m(A_n)\,.$$

Now $\bigcup_{n=1}^{\infty} A_1 \setminus A_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$, so

$$m\left(\bigcup_{n=1}^{\infty} A_1 \setminus A_n\right) = m(A_1 \setminus \bigcap_{n=1}^{\infty} A_n) = m(A_1) - m(\bigcap_{n=1}^{\infty} A_n)$$

(using that $m(A_1) < \infty$ again).

Combining the previous two displayed equations gives

$$m(A_1) - \lim_{n\to\infty} m(A_n) = m(A_1) - m(\bigcap_{n=1}^{\infty} A_n)\,,$$

hence $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$ as required.

(f) We might choose $A_n = [n, \infty)$. Now $m(A_n) = \infty$ for each n, so

$$\lim_{n\to\infty} m(A_n) = \infty \neq 0 = m(\emptyset) = m(\bigcap_{n=1}^{\infty} A_n)\,.$$

Question 3 (a) $f : \mathbb{R} \to \mathbb{R}$ is *(Lebesgue-) measurable* if for every interval I, the set $f^{-1}(I)$ is measurable. [There are various well-known equivalent definitions - any of these will also score full marks]

(b) A (non-negative) function $\varphi : \mathbb{R} \to \mathbb{R}$ is *simple* if it is measurable and only takes finitely many values, i.e. if $\varphi(\mathbb{R}) = \{a_1, \dots, a_n\}$ for some $a_1, \dots, a_n \in \mathbb{R}$.

(c) If $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ then

$$\int_E \varphi \, dm = \sum_{i=1}^n a_i m(A_i \cap E) \, .$$

(d)

$$\int_E f \, dm = \sup_\varphi \int_E \varphi \, dm \, ,$$

where the supremum is taken over those simple functions φ satisfying $0 \leq \varphi(x) \leq f(x)$ for all $x \in E$.

(e) If φ is a simple function satisfying $0 \leq \varphi(x) \leq f(x)$ for all $x \in E$, then also $0 \leq \varphi(x) \leq g(x)$ for all $x \in E$. So the supremum in the definition of $\int_E g \, dm$ is taken over a larger collection of simple functions than the one in the definition of $\int_E f \, dm$, hence $\int_E f \, dm \leq \int_E g \, dm$.

(f) [4 marks] If $f = 0$ almost everywhere, and $0 \leq \varphi \leq f$ is simple, then $\varphi = 0$ almost everywhere, since neither f nor φ take negative values. Thus $\int_{\mathbb{R}} \varphi \, dm = 0$ for all such φ, and hence $\int_{\mathbb{R}} f \, dm = 0$.

[5 marks] Conversely, if $\int_{\mathbb{R}} f \, dm = 0$ then set $E := \{x \in \mathbb{R} : f(x) > 0\}$; we wish to show that $m(E) = 0$. For each $n \geq 1$ define

$$E_n := f^{-1}([\frac{1}{n}, \infty)) \, ,$$

so that $E = \bigcup_{n=1}^\infty E_n$.

Now $\varphi_n := \frac{1}{n} \chi_{E_n}$ is simple, and $\varphi_n \leq f$ by definition of E_n, so

$$m(E_n) \ = \ n \int_{\mathbb{R}} \varphi_n \, dm \ \leq \ n \int_{\mathbb{R}} f \, dm = 0 \, ,$$

hence $m(E) = m(\bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty m(E_n) = 0$, as required.

Question 4 (a) Fatou's Lemma states that if $\{f_n\}$ is a sequence of non-negative measurable functions (defined on a measurable set E), then

$$\liminf_{n\to\infty} \int_E f_n\, dm \;\geq\; \int_E (\liminf_{n\to\infty} f_n)\, dm\,.$$

(b) The Monotone Convergence Theorem asserts that if $\{f_n\}$ is a sequence of non-negative measurable functions (defined on a measurable set E), and the sequence $\{f_n(x)\}_{n=1}^{\infty}$ increases monotonically to $f(x)$ for each $x \in E$, then

$$\lim_{n\to\infty} \int_E f_n\, dm \;=\; \int_E f\, dm\,.$$

(c) Now $f_n \leq f$, so $\int_E f_n\, dm \leq \int_E f\, dm$, hence

$$\limsup_{n\to\infty} \int_E f_n\, dm \leq \int_E f\, dm\,.$$

Fatou's Lemma gives

$$\int_E f\, dm \;\leq\; \liminf_{n\to\infty} \int_E f_n\, dm\,,$$

and clearly also

$$\liminf_{n\to\infty} \int_E f_n\, dm \;\leq\; \limsup_{n\to\infty} \int_E f_n\, dm\,.$$

Combining the above 3 inequalities gives

$$\int_E f\, dm = \liminf_{n\to\infty} \int_E f_n\, dm = \limsup_{n\to\infty} \int_E f_n\, dm\,,$$

so the sequence $\int_E f_n\, dm$ converges to $\int_E f\, dm$, as required.

(d) The Dominated Convergence Theorem asserts the following: Suppose $E \subset \mathbb{R}$ is measurable. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions (defined on E) such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and almost every $x \in E$, for some function g which is integrable over E. If $f = \lim_{n\to\infty} f_n$ almost everywhere then f is integrable over E, and

$$\lim_{n\to\infty} \int_E f_n\, dm \;=\; \int_E f\, dm\,.$$

(e) For each $n \geq 1$ and $x \in [1,\infty)$ we have

$$|f_n(x)| = f_n(x) = \frac{\sqrt{x}}{1+nx^4} \leq \frac{\sqrt{x}}{nx^4} \leq x^{-7/2} =: g(x)\,.$$

Since g is integrable on $[1,\infty)$, and $f_n \to 0$ pointwise, the Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \int_1^{\infty} f_n\, dm = \int_1^{\infty} 0\, dm = 0\,.$$

END OF EXAMINATION