

Solutions 1

1 First note that $m^*(B) \leq m^*(A \cup B)$, by monotonicity (Proposition 2.3), because $B \subset A \cup B$. Now $m^*(A \cup B) \leq m^*(A) + m^*(B)$ by sub-additivity (Theorem 2.5), so in fact $m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B)$. But $m^*(A) + m^*(B) = m^*(B)$ because $m^*(A) = 0$, so overall we have $m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$. Since lefthand and righthand sides of this inequality are identical, it follows that we must have equality throughout, i.e. $m^*(B) = m^*(A \cup B) = m^*(A) + m^*(B) = m^*(B)$, as required.

2 Note that $A \subset B \cup (A \Delta B)$ (as is easily checked), so $m^*(A) \leq m^*(B \cup (A \Delta B)) \leq m^*(B) + m^*(A \Delta B) = m^*(B)$, by monotonicity (Proposition 2.3), sub-additivity (Theorem 2.5), and the fact that $m^*(A \Delta B) = 0$.

Now we can reverse the roles of A and B , noting that $B \subset A \cup (A \Delta B)$ and therefore $m^*(B) \leq m^*(A \cup (A \Delta B)) \leq m^*(A) + m^*(A \Delta B) = m^*(A)$.

Combining these two inequalities gives $m^*(A) = m^*(B)$, as required.

3 If a system $\{I_n\}_{n=1}^{\infty}$ of intervals covers A , then the translated intervals $\{I_n + t\}_{n=1}^{\infty}$ clearly cover $A + t$, and the total lengths of the systems are equal, i.e. $\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(I_n + t)$, because $l(I_n) = l(I_n + t)$ for each n .

Conversely, if the system $\{J_n\}_{n=1}^{\infty}$ of intervals covers $A + t$, then the translated intervals $\{J_n - t\}_{n=1}^{\infty}$ cover A , and the total lengths of the systems are equal, i.e. $\sum_{n=1}^{\infty} l(J_n) = \sum_{n=1}^{\infty} l(J_n - t)$, because $l(J_n) = l(J_n - t)$ for each n .

So we have a one-to-one correspondence between coverings (by intervals) of A and $A + t$, and this correspondence preserves the total length of the covering. It follows that the sets Z_A and Z_{A+t} are identical, hence $m^*(A) = \inf Z_A = \inf Z_{A+t} = m^*(A + t)$, as required.

4 There are many possible answers to this question. One collection consists of \mathbb{R} itself, together with every countable subset of \mathbb{R} .

5 First suppose x and y belong to exactly the same subsets in \mathcal{F} . In other words, if $A \in \mathcal{F}$ then $x \in A$ if and only if $y \in A$. So $\delta_x(A) = \delta_y(A)$ for every $A \in \mathcal{F}$. Therefore $\delta_x = \delta_y$.

Conversely, suppose that $\delta_x = \delta_y$. This means that $\delta_x(A) = \delta_y(A)$ for every $A \in \mathcal{F}$. But this means that x and y belong to exactly the same subsets in \mathcal{F} .

6 Without loss of generality we may suppose that f is monotone increasing. If f is increasing then for each $a \in \mathbb{R}$, the set $f^{-1}([a, \infty))$ is either empty (if $a > f(x)$ for all $x \in \mathbb{R}$), or the interval $[b, \infty)$ or (b, ∞) for some $b (= \inf\{x \in \mathbb{R} : f(x) \geq a\})$. These are all measurable sets, so f is measurable.

7 Since f is measurable, $f^{-1}(I)$ is a measurable set for any choice of interval I . In particular, we may choose $I = [c, c] = \{c\}$, and see that $\{x : f(x) = c\} = f^{-1}(\{c\}) = f^{-1}([c, c])$ is measurable for every $c \in \mathbb{R}$, as required.

8 Given $f : \mathbb{R} \rightarrow \mathbb{R}$, show that the condition “ $\{x : f(x) = c\}$ is measurable for all $c \in \mathbb{R}$ ” is NOT enough to guarantee that f is measurable.

Let A be a non-measurable set contained in $(0, 1)$ (for example the set described in the Appendix of Capinski & Kopp). Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x$ for $x \in A$, and $f(x) = -x$ for $x \in [0, 1] \setminus A$, and $f(x) = 5$ for $x \in (-\infty, 0) \cup (1, \infty)$.

If $c \in A$ then $\{x : f(x) = c\} = \{c\}$, and the singleton $\{c\}$ is a measurable set. If $c \in [-1, 0] \setminus -A$ then $\{x : f(x) = c\} = \{-c\}$, again a measurable set. Now if $c = 5$ then $\{x : f(x) = c\} = (-\infty, 0) \cup (1, \infty)$, again a measurable set. For any other c we have $\{x : f(x) = c\} = \emptyset$, again a measurable set.

So $\{x : f(x) = c\}$ is a measurable set for every $c \in \mathbb{R}$.

However, $f^{-1}(0, 1) = A$, a non-measurable set, so f is a non-measurable function.

9 Give an example of a non-measurable function f such that f^2 is measurable (where we define f^2 by $f^2(x) = f(x)^2$).

Let A be a non-measurable set. Define the function f by $f(x) = 1$ if $x \in A$ and $f(x) = -1$ if $x \notin A$. Then $f^{-1}[1, 1] = f^{-1}(1) = A$ is non-measurable, so f is non-measurable. But f^2 is the constant function identically equal to 1, hence measurable.

10 Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable. Show that its derivative f' is a measurable function.

By definition of derivative, $f'(x) = \lim_{\delta \rightarrow 0} \frac{f(x+\delta) - f(x)}{\delta}$. In particular,

$$f'(x) = \lim_{n \rightarrow \infty} \frac{f(x + 1/n) - f(x)}{1/n} = \lim_{n \rightarrow \infty} f_n(x),$$

where we define the sequence of functions $\{f_n\}_{n=1}^{\infty}$ by $f_n(x) = n(f(x + 1/n) - f(x))$. Each f_n is a measurable function (since for each n , the function $x \mapsto f(x + 1/n)$ is easily seen to be measurable), so by Theorem 3.5, $f' = \lim_{n \rightarrow \infty} f_n$ is measurable.