Queen Mary University of London

MTH716U / MTHM007 Measure Theory & Probability

Solutions 1

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1 *First note that $m^*(B) \leq m^*(A \cup B)$, by monotonicity (Proposition 2.3), because $B \subset A \cup B$. Now $m^*(A \cup B) \leq m^*(A) + m^*(B)$ by sub-adddivity (Theorem 2.5), so in fact $m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B)$. But $m^*(A) + m^*(B) = m^*(B)$ because $m^*(A) = 0$, so overall we have $m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$. Since lefthand and righthand sides of this inequality are identical, it follows that we must have equality throughout, i.e. $m^*(B) = m^*(A \cup B) = m^*(A) + m^*(B) = m^*(B)$, as required.*

2 *Note that $A \subset B \cup (A\Delta B)$ (as is easily checked), so $m^*(A) \leq m^*(B \cup (A\Delta B)) \leq m^*(B) + m^*(A\Delta B) = m^*(B)$, by monotonicity (Proposition 2.3), sub-additivity (Theorem 2.5), and the fact that $m^*(A\Delta B) = 0$.*

Now we can reverse the roles of A and B, noting that $B \subset A \cup (A\Delta B)$ and therefore $m^*(B) \leq m^*(A \cup (A\Delta B)) \leq m^*(A) + m^*(A\Delta B) = m^*(A)$.

Combining these two inequalities gives $m^*(A) = m^*(B)$, as required.

3 *If a system $\{I_n\}_{n=1}^\infty$ of intervals covers A, then the translated intervals $\{I_n + t\}_{n=1}^\infty$ clearly cover $A + t$, and the total lengths of the systems are equal, i.e. $\sum_{n=1}^\infty l(I_n) = \sum_{n=1}^\infty l(I_n + t)$, because $l(I_n) = l(I_n + t)$ for each n.*

Conversely, if the system $\{J_n\}_{n=1}^\infty$ of intervals covers $A + t$, then the translated intervals $\{J_n - t\}_{n=1}^\infty$ cover A, and the total lengths of the systems are equal, i.e. $\sum_{n=1}^\infty l(J_n) = \sum_{n=1}^\infty l(J_n - t)$, because $l(J_n) = l(J_n - t)$ for each n.

So we have a one-to-one correspondence between coverings (by intervals) of A and $A + t$, and this correspondence preserves the total length of the covering. It follows that the sets Z_A and Z_{A+t} are identical, hence $m^*(A) = \inf Z_A = \inf Z_{A+t} = m^*(A + t)$, as required.

4 *There are many possible answers to this question. One collection consists of $\mathbb{R}$ itself, together with every countable subset of $\mathbb{R}$.*

5 *First suppose x and y belong to exactly the same subsets in $\mathscr{F}$. In other words, if $A \in \mathscr{F}$ then $x \in A$ if and only if $y \in A$. So $\delta_x(A) = \delta_y(A)$ for every $A \in \mathscr{F}$. Therefore $\delta_x = \delta_y$.*

Conversely, suppose that $\delta_x = \delta_y$. This means that $\delta_x(A) = \delta_y(A)$ for every $A \in \mathscr{F}$. But this means that x and y belong to exactly the same subsets in $\mathscr{F}$.

6 *Without loss of generality we may suppose that f is monotone increasing. If f is increasing then for each $a \in \mathbb{R}$, the set $f^{-1}([a,\infty))$ is either empty (if $a > f(x)$ for all $x \in \mathbb{R}$), or the interval $[b,\infty)$ or (b,∞) for some $b (= \inf\{x \in \mathbb{R} : f(x) \geq a\})$. These are all measurable sets, so f is measurable.*

7 *Since* f *is measurable,* $f^{-1}(I)$ *is a measurable set for any choice of interval* I*. In particular, we may choose* $I = [c,c] = \{c\}$*, and see that* $\{x : f(x) = c\} = f^{-1}(\{c\}) = f^{-1}([c,c])$ *is measurable for every* $c \in \mathbb{R}$*, as required.*

8 *Given* $f : \mathbb{R} \to \mathbb{R}$*, show that the condition "*$\{x : f(x) = c\}$ *is measurable for all* $c \in \mathbb{R}$*" is* NOT *enough to guarantee that* f *is measurable.*

Let A *be a non-measurable set contained in* $(0,1)$ *(for example the set described in the Appendix of Capinski & Kopp). Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x$ *for* $x \in A$*, and* $f(x) = -x$ *for* $x \in [0,1] \setminus A$*, and* $f(x) = 5$ *for* $x \in (-\infty,0) \cup (1,\infty)$*.*

If $c \in A$ *then* $\{x : f(x) = c\} = \{c\}$*, and the singleton* $\{c\}$ *is a measurable set. If* $c \in [-1,0] \setminus -A$ *then* $\{x : f(x) = c\} = \{-c\}$*, again a measurable set. Now if* $c = 5$ *then* $\{x : f(x) = c\} = (-\infty,0) \cup (1,\infty)$*, again a measurable set. For any other* c *we have* $\{x : f(x) = c\} = \emptyset$*, again a measurable set.*

So $\{x : f(x) = c\}$ *is a measurable set for every* $c \in \mathbb{R}$*.*

However, $f^{-1}(0,1) = A$*, a non-measurable set, so* f *is a non-measurable function.*

9 *Give an example of a non-measurable function* f *such that* f^2 *is measurable (where we define* f^2 *by* $f^2(x) = f(x)^2$*).*

Let A *be a non-measurable set. Define the function* f *by* $f(x) = 1$ *if* $x \in A$ *and* $f(x) = -1$ *if* $x \notin A$*. Then* $f^{-1}[1,1] = f^{-1}(1) = A$ *is non-measurable, so* f *is non-measurable. But* f^2 *is the constant function identically equal to 1, hence measurable.*

10 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable. Show that its derivative* f' *is a measurable function.*

By definition of derivative, $f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta)-f(x)}{\delta}$*. In particular,*

$$f'(x) = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} = \lim_{n\to\infty} f_n(x)\,,$$

where we define the sequence of functions $\{f_n\}_{n=1}^{\infty}$ *by* $f_n(x) = n\,(f(x+1/n) - f(x))$*. Each* f_n *is a measurable function (since for each* n*, the function* $x \mapsto f(x+1/n)$ *is easily seen to be measurable), so by Theorem 3.5,* $f' = \lim_{n\to\infty} f_n$ *is measurable.*